Real algebraic curves in real minimal del Pezzo surfaces

Ph.D. Defense

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École Polytechnique

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History and context

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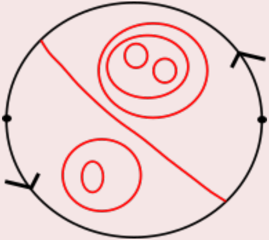
Example real scheme: $\mathcal{J} \sqcup \langle 1 \rangle \sqcup \langle\langle 2 \rangle\rangle$



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- Constructions real algebraic curves with prescribed topology

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Example of Bézout-type restrictions for $d = 4$ real plane curves

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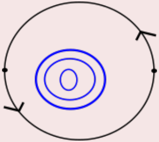

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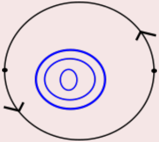

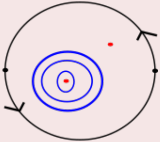

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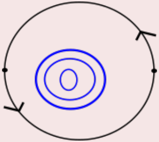
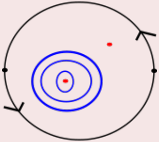
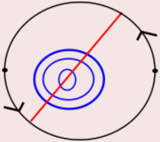
$\langle\langle 1 \rangle\rangle$   

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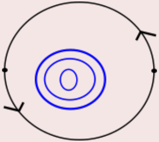

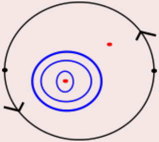

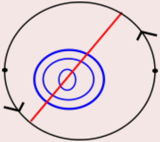

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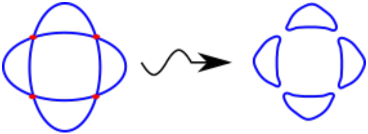

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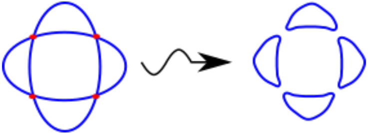

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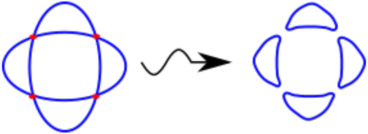

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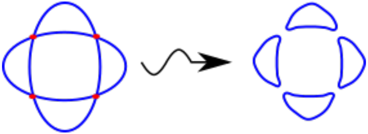

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- Up to now, patchworking and its generalizations are some of the most powerful construction methods

Constructions in general

How construct real algebraic curves in

- real non-toric surfaces or
- in toric surfaces with real structure non-compatible with torus action?

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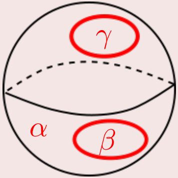

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- What about $d = 5$?

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$\forall\ 0 \leq l \leq 17$, every real scheme in bidegree $(5,5)$ with l ovals, that is not previously prohibited, is realizable by a non-separating (and/or separating) real algebraic curve of bidegree $(5,5)$ on the quadric ellipsoid.

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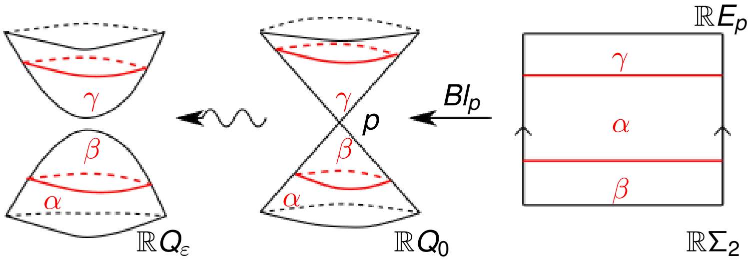
- Main strategy: reduce ourselves to construct real curves on Σ_2 degenerating the quadric ellipsoid to the quadratic cone in $\mathbb{C}P^3$

Construction via example: $\alpha \sqcup \langle\beta\rangle \sqcup \langle\gamma\rangle$



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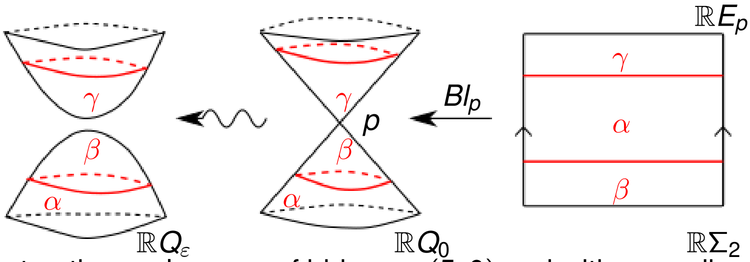

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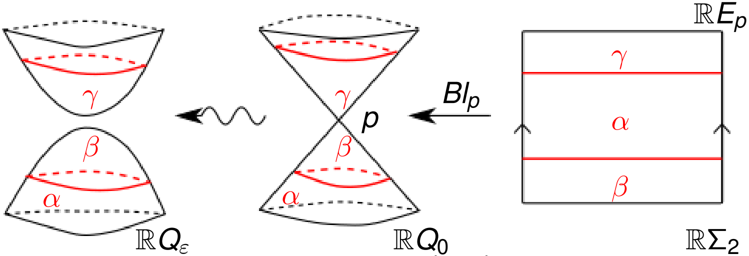

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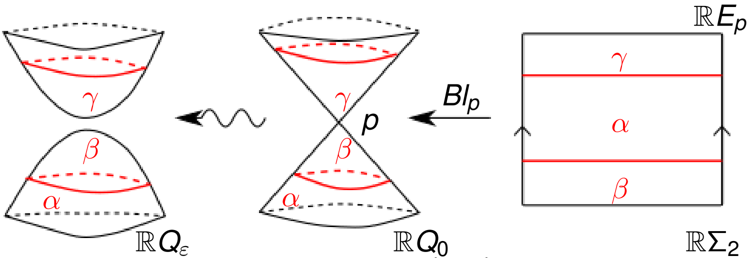

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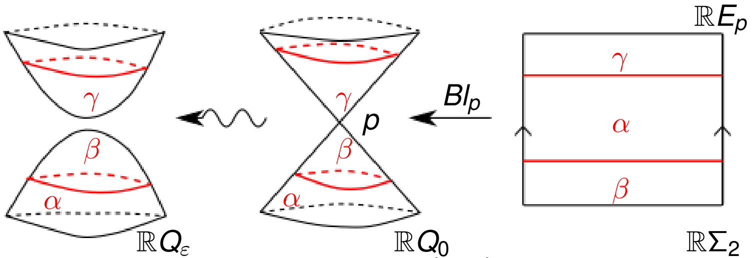

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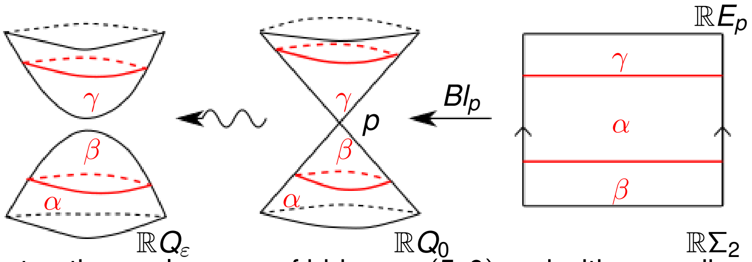

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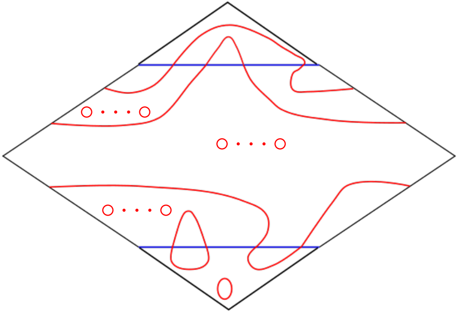

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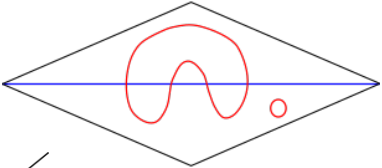

Chart of bidigree $(2, 0)$
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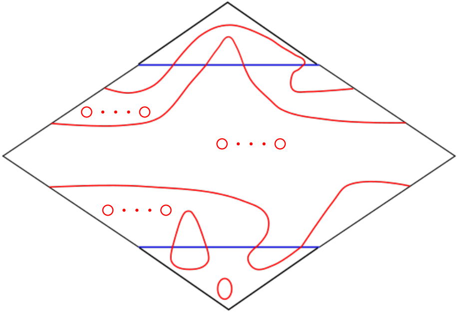

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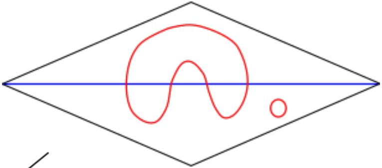

Chart of bidigree $(2,0)$
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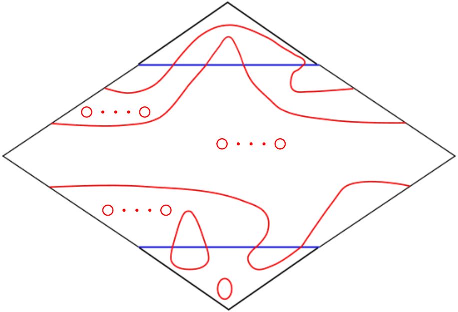

Chart of bidegree $(5,0)$
real curve in $\mathbb{R}\Sigma_2$

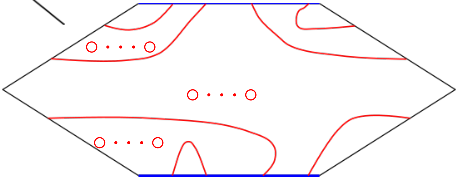

Chart of bidegree $(3,4)$
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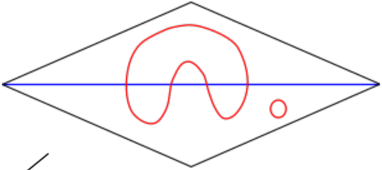

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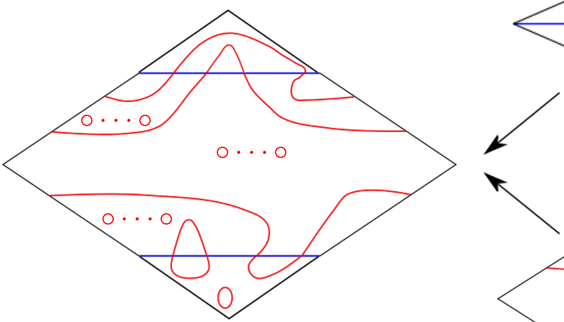

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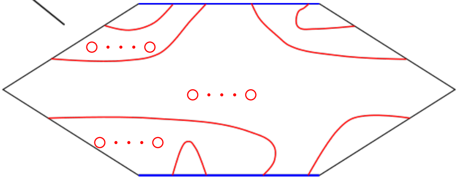

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Real algebraic curves on real minimal del Pezzo surfaces

- Real minimal rational surfaces (Comesatti ’13-’14)

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- Anti-(bi)canonical map exhibits such surfaces as ramified double covers

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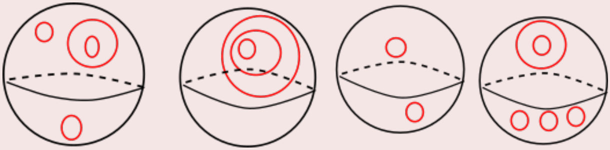
- ovals

Example of a real scheme on $\mathbb{R}X$: $2 \sqcup \langle 1 \rangle : 1 \sqcup \langle 1 \rangle : 2 : 3 \sqcup \langle 1 \rangle$



X_1 X_2 X_3 X_4

4-spheres real del Pezzo surfaces of degree 2

Harnack-Klein inequality + Adjunction formula:

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Bound on number of ovals for a class d real curve

$$l \leq d(d-1)+2$$

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Every real scheme in class $d = 1, 2$, non-prohibited by Harnack-Klein, is realizable by a real curve of class d in X

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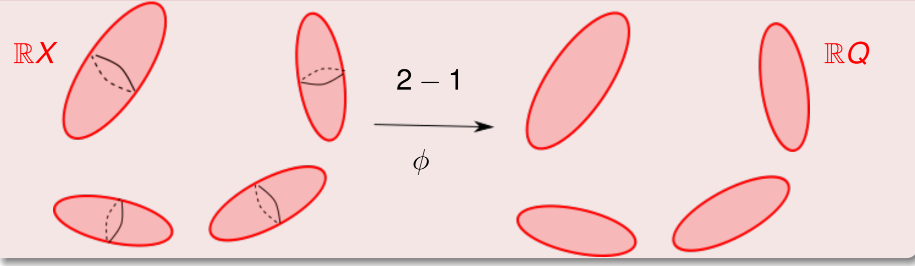
Real schemes in class 1:

- $1:1:0:0$
- $1:0:0:0$
- $0:0:0:0$
- $2:0:0:0$

4-spheres real del Pezzo surfaces of degree 2

Anti-canonical map: $\phi : X \xrightarrow{2-1} \mathbb{C}P^2 \hookleftarrow Q$



- $Q \subset \mathbb{C}P^2$ a real non-singular maximal quartic

4-spheres real del Pezzo surfaces of degree 2

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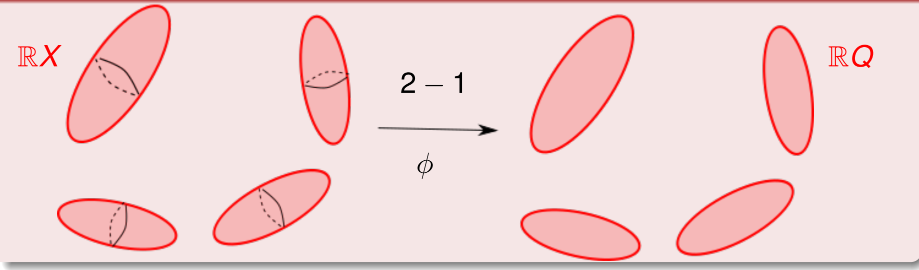

- $Q \subset \mathbb{C}P^2$ a real non-singular maximal quartic
- the double cover of $\mathbb{C}P^2$ ramified along Q yields a 4-spheres real del Pezzo surface

4-spheres real del Pezzo surfaces of degree 2

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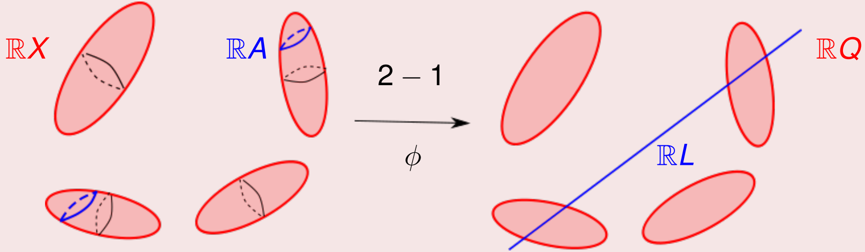

$A \subset X$ is real of class 1

4-spheres real del Pezzo surfaces of degree 2

Real schemes in class 1:

- $1 : 1 : 0 : 0$ ✓
- $1 : 0 : 0 : 0$
- $0 : 0 : 0 : 0$
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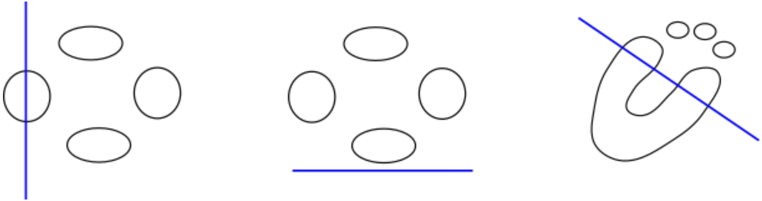

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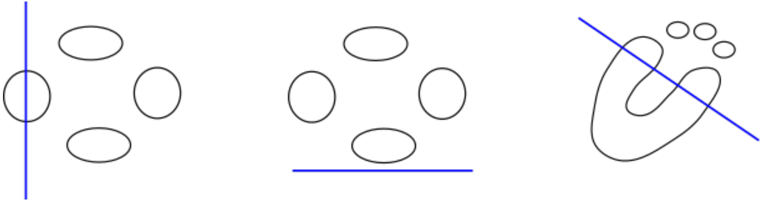

Via ϕ, realization of all real schemes in class 1 and 2

4-spheres real del Pezzo surfaces of degree 2

Class $d \geq 3$

- Harnack-Klein inequality no complete set of restrictions

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- New restrictions (Welschinger-type invariants)

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New

- New restrictions (Welschinger-type invariants)
- New construction method breaking the symmetry (Degeneration and Patchworking)

4-spheres real del Pezzo surfaces of degree 2

Welschinger invariants:

- real analogues of $g = 0$ Gromov-Witten invariants (Welschinger '05)

4-spheres real del Pezzo surfaces of degree 2

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Case of X:

- $g = 0$ (Itenberg, Kharlamov and Shustin '17)
- generalizations to higher genus $0 < g \leq 3$ (Shustin '14)

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Key point: Exploit the existence of $g = 3$ interpolating real curves

Existence of interpolating real algebraic curves of class k and genus 3

- $k \in \mathbb{Z}_{>1}$, odd $r_1, r_2 \in \mathbb{Z}_{>0}$ such that $r_1 + r_2 = 2k$
- $\mathcal{P}$ be a generic real configuration of $2k + 2$ points on X

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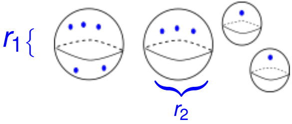
- $X_i \supset r_i$ points, with $i = 1, 2$
- X_3 and X_4 both contain one point



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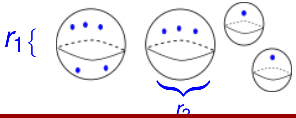


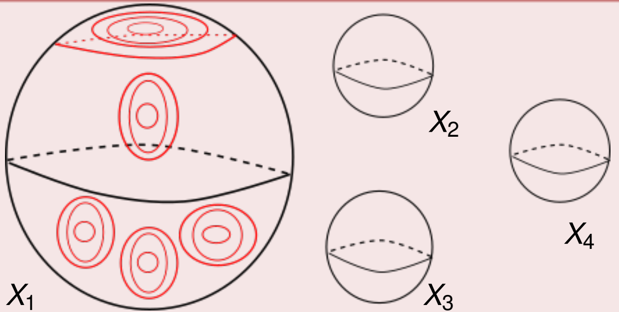
Shustin '14

$\exists$ a real curve T of class k and $g = 3$ in X through $\mathcal{P}$. The points of $\mathcal{P}$ belong to the one-dimensional connected components of $\mathbb{R}T$

4-spheres real del Pezzo surfaces of degree 2

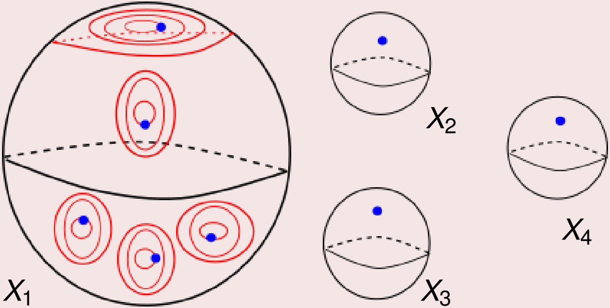
Assume ∃ A of class 5 in X realizing the following real scheme

Example of unrealizability of a real scheme in class 5:
$\bigsqcup_4 \langle\langle 1 \rangle\rangle \ \sqcup \ \langle\langle\langle 1 \rangle\rangle\rangle : 0 : 0 : 0$



- $k = 3$, $r_1 = 5$ and $r_2 = 1$.
- Choose 8 points on $\mathbb{R}X$ as below

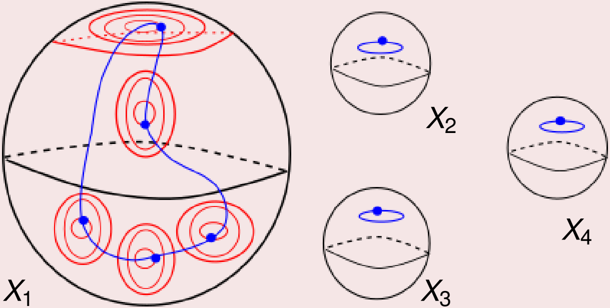
Example class 5:



4-spheres real del Pezzo surfaces of degree 2

- $\exists T$ of class 3 and $g = 3$ through the 8 points
- The number of connected components of $\mathbb{R}T$ is 4
- $A \circ T = 30$

Example class 5: $\mathbb{R}A \circ \mathbb{R}T = 32 > 30$



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Assume $\mathbb{R}A$ lies on $t \leq 4$ spheres of $\mathbb{R}X$ and it has

- $N_1,..,N_{r_1}$ disjoint nests of depth j_h^1 on X_1

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Proposition (M)

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Proposition (M)

1. If $r_1 = 2k - 1$ and $r_2 = 1$, then $\sum_{h=1}^{r_1} j_h^1 \leq dk - (t-1)$
2. If $r_1, r_2 > 1$, then $\sum_{h=1}^{r_1} j_h^1 + \sum_{h=r_1+1}^{r_1+r_2} j_h^2 \leq dk - (t-2)$

4-spheres real del Pezzo surfaces of degree 2

Theorem (M)

- $\exists$ 74 real schemes in class 3, with 8 ovals, non-prohibited by Harnack-Klein inequality and new restrictions
- 47 among those are realizable in class 3

4-spheres real del Pezzo surfaces of degree 2

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Remark 2

Real scheme $2 \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle$ realizable by a real symplectic curve. Not known if realizable by a real algebraic curve

4-spheres real del Pezzo surfaces of degree 2

- Class 3: Realization of $\sim$ 15 real schemes via the map ϕ

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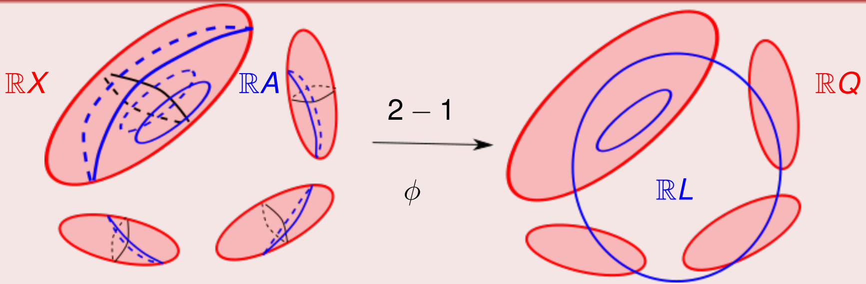
Symmetry via $\phi : X \xrightarrow{2-1} \mathbb{C}P^2 \hookleftarrow Q$.



Symmetric curves and plane curves

$$\left\{ \text{Symmetric class } d \text{ curves in } X \right\} \longleftarrow \left\{ \text{Degree } d \text{ plane curves} \right\}$$

Breaking the symmetry

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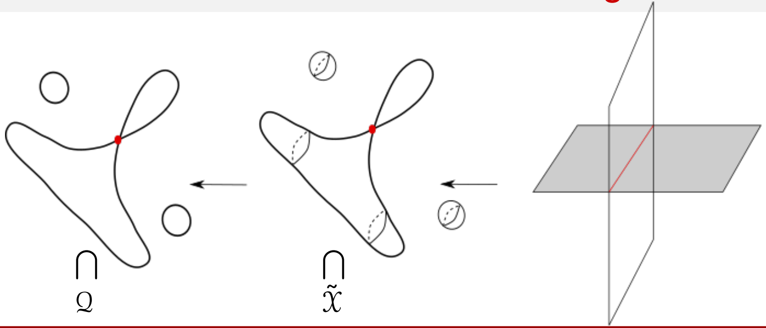
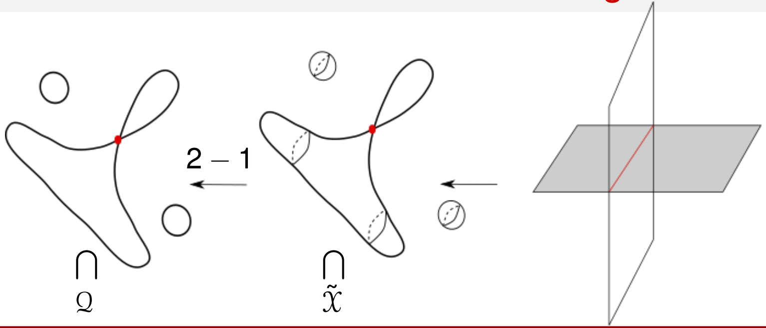
Remark

Some real schemes, if realizable, can not be realized via the anti-canonical map. We can realize some of them for every class $d \geq 5$.



Degeneration

4-spheres real del Pezzo surfaces of degree 2



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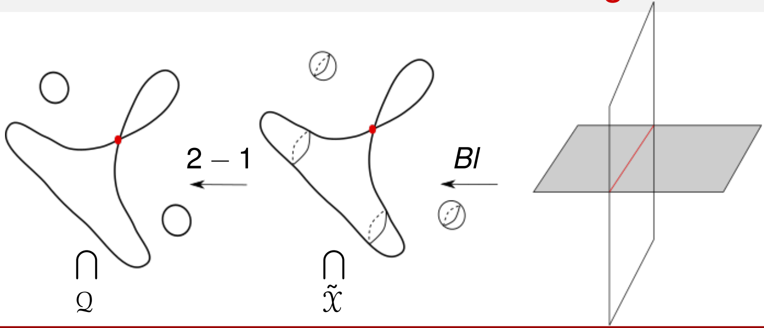

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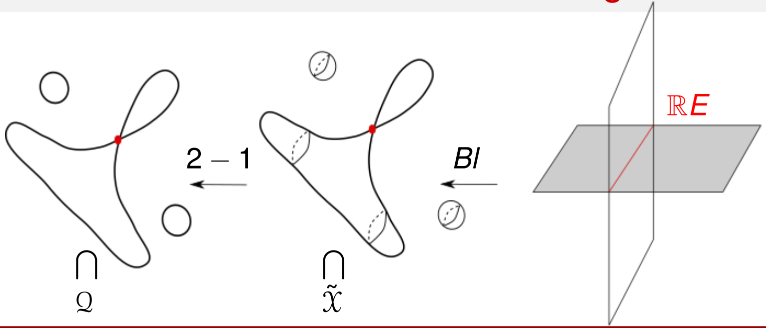

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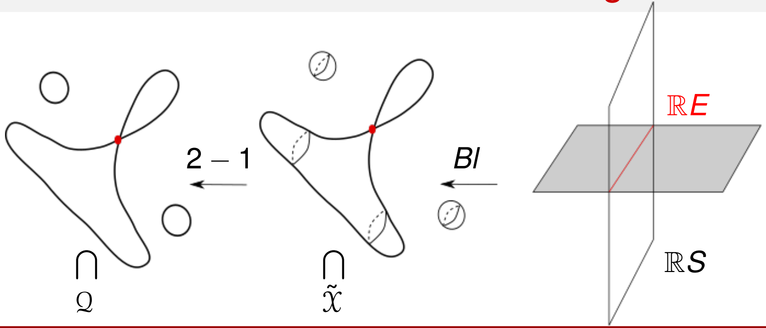

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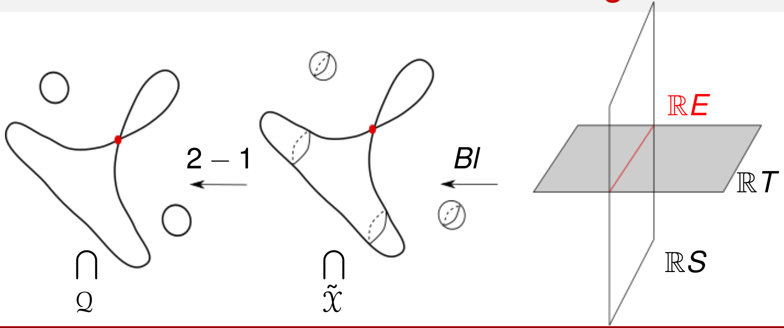

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- S is the quadric ellipsoid, $E \subset S$ bidegree $(1,1)$ curve; (T, E) is a nodal degree 2 del Pezzo pair, $E \subset T$ (-2)-curve

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Cut along $\mathbb{R}E$ and glue back the surfaces into $\bigsqcup_{j=1}^{4} S^2$

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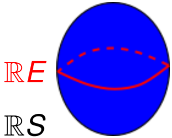

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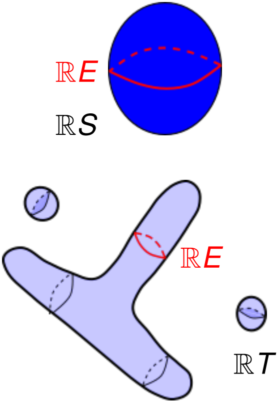

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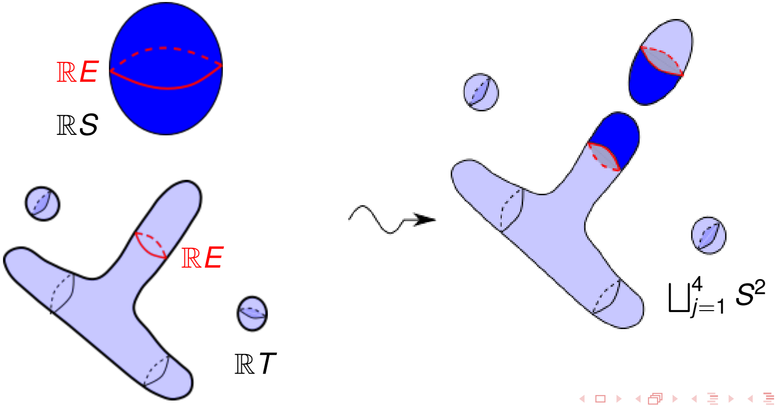

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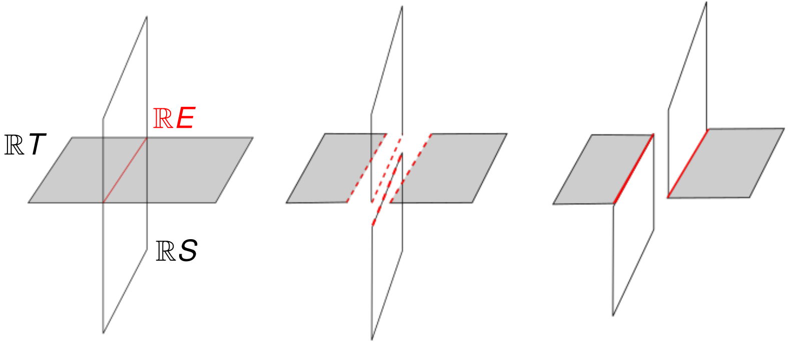
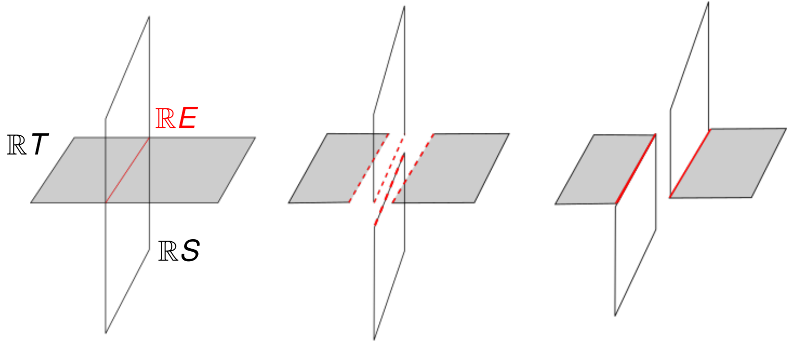


Topological operation

4-spheres real del Pezzo surfaces of degree 2



Topological operation

- Either choices of gluing give us $\bigsqcup_{j=1}^{4} S^2$.

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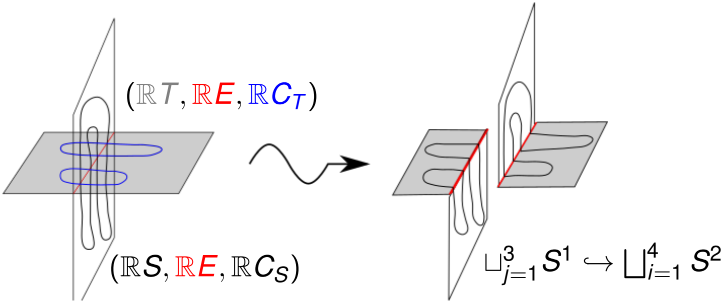

Topological operation via an example

- Real algebraic curves $C_S \subset S$ and $C_T \subset T$
- $C_S \pitchfork C_T$ in 4 real points $\subset E$

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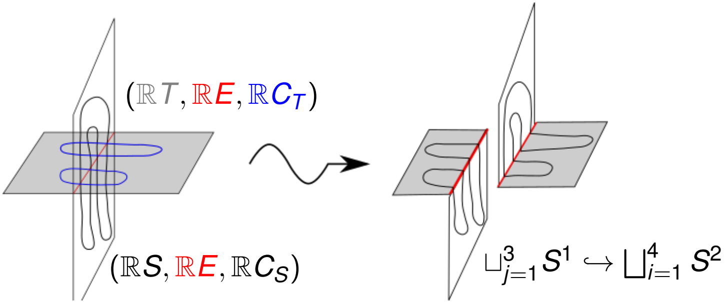

Topological operation via an example

- Real algebraic curves $C_S \subset S$ and $C_T \subset T$
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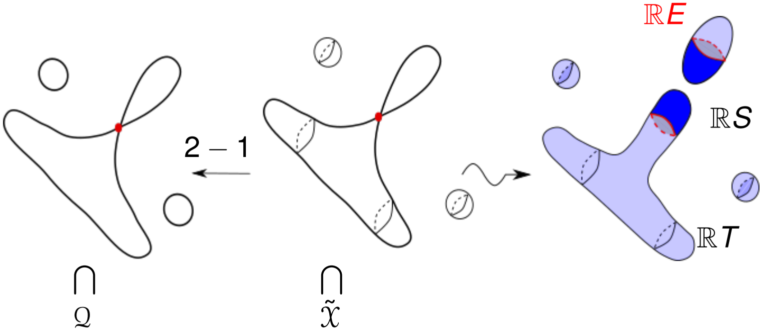
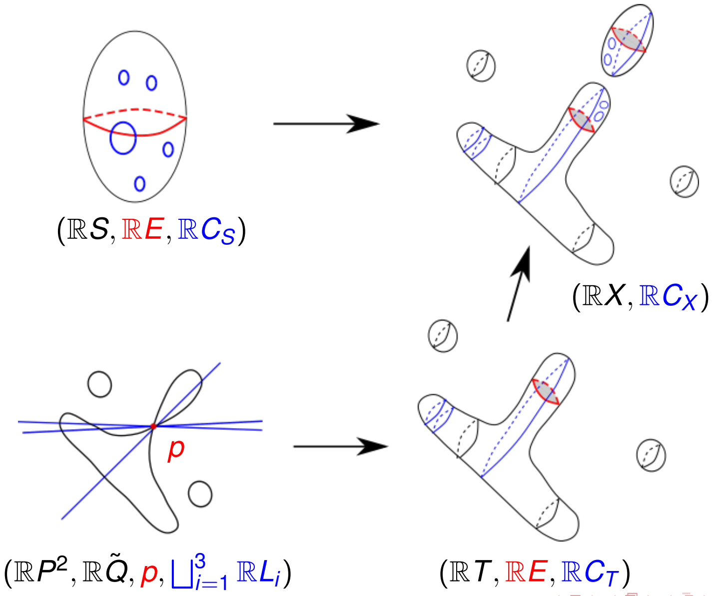
- **Such topological construction is realizable algebraically!** (Shustin and Tyomkin '06)

4-spheres real del Pezzo surfaces of degree 2



Patchworking of surfaces

Realization of class 3 real scheme: $\langle 1 \rangle \sqcup \langle 2 \rangle : 3 : 0 : 0$



Real minimal del Pezzo surfaces of degree 1

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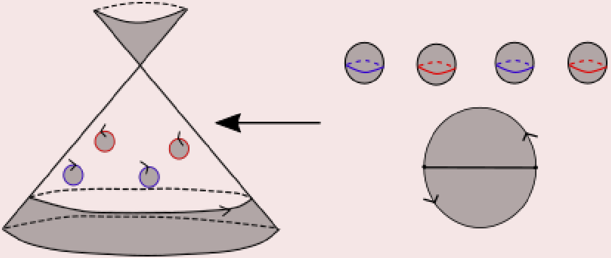
Anti-bicanonical map: $\psi : Y \xrightarrow{2-1} Q \hookleftarrow \tilde{S}$



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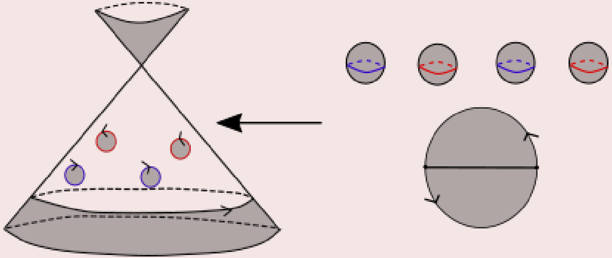

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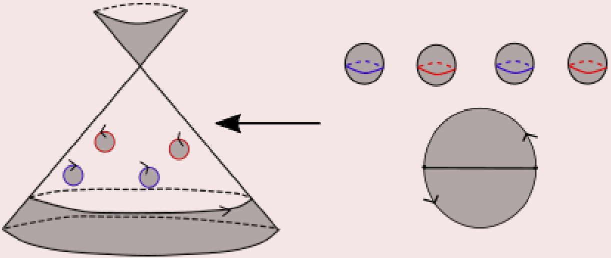

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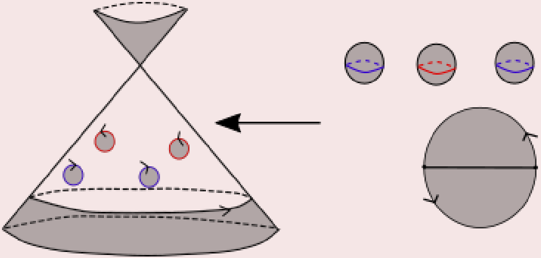

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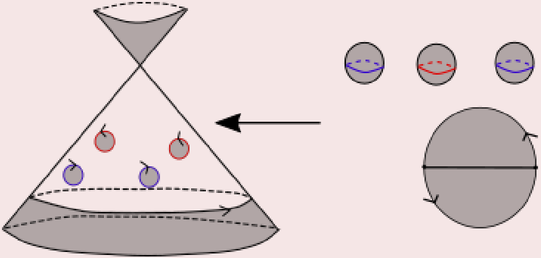

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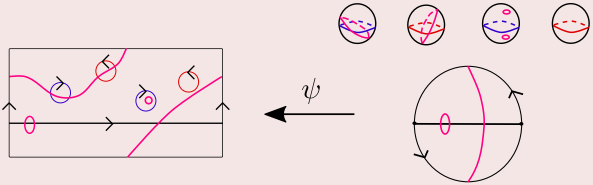
Example



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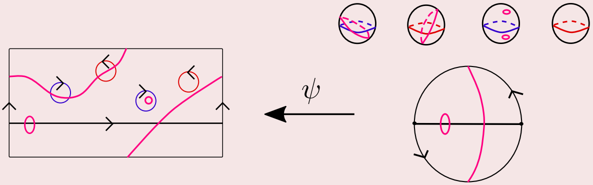

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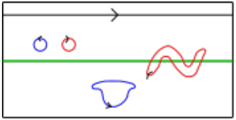

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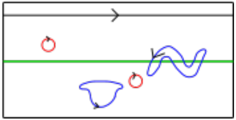

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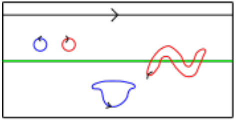

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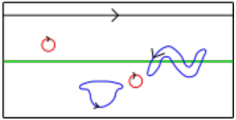

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- Expect more difficulties for classifications in class $d \geq 4$

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- Explore algebraic vs symplectic on 4-spheres real del Pezzo surfaces of degree 2

THANK YOU FOR THE ATTENTION!