BETTI NUMBERS OF REAL SEMISTABLE DEGENERATIONS VIA REAL LOGARITHMIC GEOMETRY

EMILIANO AMBROSI AND MATILDE MANZAROLI

ABSTRACT. Let $X \to C$ be a totally real semistable degeneration over a smooth real curve C with degenerate fiber X_0. Assuming that the irreducible components of X_0 are simple from a cohomological point of view, we give a bound for the individual Betti numbers of a real smooth fiber near 0 in terms of the complex geometry of the degeneration. This generalizes previous work of Renaudineau-Shaw, obtained via combinatorial techniques, for tropical degenerations of hypersurfaces in smooth toric varieties. The main new ingredient is the use of real logarithmic geometry, which allows to work with not necessarily toric degenerations.

CONTENTS

1. INTRODUCTION

Let X be a real algebraic variety, let $X(\mathbb{C})$ be the set of its complex points and $X(\mathbb{R})$ the set of its real points. For a topological space Y, set $b_i(Y) := \dim_{\mathbb{F}_2}(H^i(Y, \mathbb{Z}/2\mathbb{Z}))$ for its i^{th} Betti number.

1.1. **Complex semistable degeneration.** A classical tool to study irreducible smooth projective varieties is to degenerate them to a union of irreducible simpler varieties.

A celebrated theorem of Steenbrink ([Ste76]) shows that if C is a smooth complex curve and $X \to C$ is a semistable degeneration of projective varieties with singular fiber X_0, then the rational cohomology of a general smooth fiber X_t can be computed from the geometry of X_0. More precisely, Steenbrink shows that, for every $q \geq 0$, there exists a complex $E_2^{q,\bullet}$ of $\mathbb{Q}$-vector spaces, depending only on X_0, such that

$$\dim(H^i(X_t, \mathbb{Q})) = \sum_{p+q=i} \dim H^p(E_2^{q,\bullet}). \tag{1.1.1}$$

The goal of this paper is to try to extend this kind of results to real semistable degenerations and to understand to which extent the topology of the real special fiber control the topology of the real general fiber.

1.2. **Real semistable degeneration.** Assume from now on that $X \to C$ is a real semistable degeneration with singular fiber X_0. Of course, one cannot expect equalities similar to (1.1.1) to hold in the real setting, since the real topology of the special fiber can drastically change in different fibers near 0.

The idea is then to try to compare the real part of the special fiber with its complex counterpart and, afterwards, to relate the latter with the real part of a fiber near 0. In order to do this, let us first recall the Smith-Thom inequality [Tho15],

$$\sum_i b_i(X(\mathbb{R})) \leq \sum_i b_i(X(\mathbb{C})) \tag{1.2.1}$$

which bounds the total Betti number of the real topology of a real variety X with the one of its complexification. This is one of the few general results comparing the real and the complex topology of real algebraic varieties. A special role, in the study of the topology of real algebraic variety, is played by *maximal* varieties, i.e. the ones for which (1.2.1) is an equality.

Inspired by (1.2.1), one could hope to obtain a bound for the topology of a real fiber near 0 in terms of data that depends only on its complexification. Recently, using tropical geometry, Renaudineau-Shaw proved ([RS23, Theorem 1.4]), confirming a conjecture of Itenberg ([Ite17]), that for a fiber X_t near 0 of a real hypersurface inside a real *toric* degeneration constructed via *primitive combinatorial Viro patchworking*, one has that

$$b_i(X_t(\mathbb{R})) \leq \sum_j h^{i,j}(X_t), \tag{1.2.2}$$

where $h^{i,j}(X) := \dim(H^i(X, \Omega^j_X))$.

While these conjectures and results were limited to combinatorial and toric situations, we go one step further giving a purely algebraic geometric setting in which an inequality close to (1.2.2) holds. Our main results (Theorem 1.3.1, Corollary 1.3.3), show that if X_0 can be stratified with components that are simple, from a cohomological point of view, then X_t satisfies the inequality (1.2.2), up to the dimension of the 2-torsion in some cohomology group. The main novelty of our approach is the use of (real) logarithmic geometry, which allows to extend previous results for toric degenerations to more general families.

Remark 1.2.3. Very recently, there have been other two generalizations of the results and the techniques in [RS23]:

- On the combinatorial side, in [BdMR22], Brugallé, López de Medrano and Rau extend the main results of [RS23] to PL-manifolds associated to non-necessarily convex triangulations of non-singular lattice polytopes.
- On the geometric side, in [RRS23], Renaudineau, Rau and Shaw generalize [RS23, Theorem 1.5] and the strategy therein to general families with smooth tropicalisation near the tropical limit inside a smooth toric variety under an additional technical assumption.

1.3. **Main result.** Assume that C is a smooth real curve and $f : X \to C$ a real projective morphism which is smooth outside a real point $0 \in C(\mathbb{R})$ and strictly-semistable around 0, in the sense that the irreducible components of X_0 are smooth and, locally analytically around 0, the family $f : X(\mathbb{C}) \to C(\mathbb{C})$ is isomorphic to the standard semistable degeneration $\mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n, T]/(x_1 \dots x_n - T)) \to \mathrm{Spec}(\mathbb{C}[T])$. Assume furthermore that $f : X \to C$ is totally real, i.e. that the irreducible components of $X_0(\mathbb{C})$ are real. Write

$$X_0 = \bigcup_{i \in I} X_i$$

for the decomposition of X_0 in irreducible components and for every subset $J \subseteq I$ set

$$X_J := \bigcap_{i \in J} X_i \quad \text{and} \quad X_J^0 := X_J \setminus \bigcup_{i \notin J} X_i.$$

Then

$$X_0 = \coprod_{J \subseteq I} X_J^0$$

is a stratification $\mathfrak{I} := \{X_J^0\}_{J \subseteq I}$ of X_0 by smooth real algebraic subvarieties. Fix a refinement $\mathfrak{Z} := \{X_\Delta^0\}$ of $\mathfrak{I}$, made of smooth real algebraic varieties; see Remark 1.3.4.

In Section 3.4 we construct, for every ring A and every $q \geq 0$ a canonical cochain complex $C^\bullet_{q,\mathfrak{Z},A}$ of A-modules depending only on the complex geometry of the stratification $\mathfrak{Z}$. Inspired by the geometric properties of degenerations constructed via primitive patchworking (see Remark 1.3.2), we consider the following conditions on the members of $\mathfrak{Z}$.

(a) $H^i(X_\Delta^0(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = 0$, for all $i \geq 1$ and $X_\Delta^0 \in \mathfrak{Z}$;
(b) X_Δ^0 is maximal, for all $X_\Delta^0 \in \mathfrak{Z}$;
(c) the mixed Hodge structure on $H^i(X_\Delta^0(\mathbb{C}), \mathbb{Q})$ is pure of type (i,i) and $H^i(X_\Delta^0(\mathbb{C}), \mathbb{Z})$ is torsion free, for all $i \geq 1$ and $X_\Delta^0 \in \mathfrak{Z}$.

Our main result is then the following.

Theorem 1.3.1.
(1) Assume that (a) and (b) hold. Then, for every $t \in C(\mathbb{R})$ close to 0 one has:

$$b_p(X_t(\mathbb{R})) \leq \sum_q \mathit{dim}(H^p(C^\bullet_{q,\mathfrak{Z},\mathbb{Z}/2\mathbb{Z}})).$$

(2) Assume that (a),(b) and (c) hold. Then for every $t \in C(\mathbb{R})$ close to 0, one has:
(i) $\mathit{dim}(H^p(C^\bullet_{q,\mathfrak{Z},\mathbb{Z}} \otimes \mathbb{Q})) = h^{p,q}(X_t)$

(ii) $C^{\bullet}_{q,3,\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z} \simeq C^{\bullet}_{q,3,\mathbb{Z}/2\mathbb{Z}}$.

Remark 1.3.2. If X_0 comes from a degeneration constructed via primitive combinatorial patchworking, then it admits a stratification made by complements of hyperplane arrangements, which satisfy the conditions (a), (b), (c) above. This shows that Theorem 1.3.1 generalizes the main result of [RS23] to a more general, non necesseraly toric, setting.

Theorem 1.3.1 directly implies the following corollary, which was the main motivation for this paper.

Corollary 1.3.3. *Assume that (a), (b), (c) hold and that $H^p(C^{\bullet}_{q,3,A})$ is torsion free for every $p, q \in \mathbb{N}$. Then for every $t \in C(\mathbb{R})$ close to 0 and every $p \in \mathbb{N}$ one has*

$$b_p(X_t(\mathbb{R})) \leq \sum_q h^{p,q}(X_t).$$

Remark 1.3.4. We point out that even in very easy examples of semistable degenerations is necessary to refine the standard stratification $\{X^0_J\}$ in order to guarantee that the strata satisfy the hypothesis in Theorem 1.3.1. If ones take the trivial degeneration of $\mathbb{P}^1$, the stratification $\{X^0_J\}$ is the trivial one and hence it does not satisfy the hypothesis (a), (b) and (c). So, one has to further stratify $\mathbb{P}^1$, for example as

$$\mathbb{P}^1 = \left(\mathbb{P}^1 - \left(\{[0,1]\} \bigsqcup \{[1,0]\}\right)\right) \coprod \{[0,1]\} \coprod \{[1,0]\},$$

which corresponds to the stratification of $\mathbb{P}^1$ into toric orbits.

Remark 1.3.5. Theorem 1.3.1 and Corollary 1.3.3 suggest two questions:

(1) Are there interesting examples of degenerations satisfying assumptions (a), (b) and (c) that cannot be constructude via the primitive patchworking?
(2) Under which conditions the $H^p(C^{\bullet}_{q,3,A})$ is torsion free?

The main difficulty in (1) is that most of the techniques developed to construct real degenerations are of toric/combinatorial nature. To try to go beyond this setting, one would need to develop completely new techniques which, while interesting, it goes far beyond the scope of this paper.

For (2), while we don't know any single example in which $H^p(C^{\bullet}_{q,3,A})$ has torsion, several intermediate questions seems already interesting. For example, if $H^p(C^{\bullet}_{q,3,A})$ has no torsion for every p, q, then $\dim(H^p(C^{\bullet}_{q,3,A})) = \dim(H^q(C^{\bullet}_{q,3,A})) = \dim(H^{n-p}(C^{\bullet}_{n-q,3,A}))$ by Theorem 1.3.1(2), Hodge-Symmetry and Poincaré duality. Can one prove this without assuming torsion freeness?

1.4. **Strategy.**

1.4.1. *Main ideas.* In order to attack this kind of problems, the general involved strategy was to separately compute complex and real information and, in a second moment, show that they are related. For example, the proof of [RS23, Theorem 1.4] involves the result in [IKMZ19] showing that tropical homology of tropical smooth varieties is related to Hodge numbers, the results [RS23, Theorem 3.7 and Remark 3.8] showing that the tropical interpretation of the primitive combinatorial Viro patchworking method ([Vir83, Vir84]) gives a way to compute the real Betti numbers via the (co)homology of a tropical sheaf on the tropical varieties, and, finally,[RS23, Section 4] which shows how to relate the (co)homology of this sheaf to tropical homology.

A different approach to related problems has been recently proposed by Brugallé in [Bru22]. There, some complex and real invariants are shown to satisfy the same gluing relations under totally real semistable degenerations, so that, in order to relate the global invariants, it is enough to relate the local invariants, which are easier to compute. Inspired by this, our basic strategy is to relate the real Betti numbers and the Hodge numbers via the geometry of a common ambient space. The main innovation of this paper is the use of real logarithmic geometry to construct and study this common ambient space, which allows to employ a more sophisticated and less combinatorial machinery. After the construction of such a space, realised in Sections 3.2.2 and 4.1.2, the cohomology of the general real fiber can be computed by a filtered complex; see Section 5.4. The idea of the use of filtered complexes is inspired by [RS23], where it was constructed via combinatorial techniques. Since these combinatorial approach are not available in our general setting, we use a different approach based on equivariant cohomology.

1.4.2. *Equivariant cohomology.* To explain our strategy in more details, let us recall a modern proof of the Smith-Thom inequality (1.2.1), as in [BBF+60, Chapter 4, IV, Pag. 55]. Let G be $\mathbb{Z}/2\mathbb{Z}$ acting via the complex conjugation on X. Then there is a spectral sequence

$$E_2^{a,b} := H^a(G, H^b(X, \mathbb{Z}/2\mathbb{Z})) \Rightarrow H_G^{a+b}(X, \mathbb{Z}/2\mathbb{Z})$$

where $H^a(G, H^b(X, \mathbb{Z}/2\mathbb{Z}))$ is the group cohomology of G acting on $H^b(X, \mathbb{Z}/2\mathbb{Z})$ and $H_G^{a+b}(X, \mathbb{Z}/2\mathbb{Z})$ is the G-equivariant cohomology of X. Since $H_G^{a+b}(X, \mathbb{Z}/2\mathbb{Z}) \simeq H^*(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ for $a + b > 2\dim(X)$, for every m strictly begger than $2\dim(X)$ one gets a filtration ${}_mF^i$ of $H^*(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ such that ${}_mF^i/{}_mF^{i+1}$ is a subquotient of $H^i(G, H^{m-i}(X, \mathbb{Z}/2\mathbb{Z}))$. Since $\dim(H^a(G, H^b(X, \mathbb{Z}/2\mathbb{Z}))) \leq \dim(H^b(X, \mathbb{Z}/2\mathbb{Z}))$, one gets the desired inequality. This argument shows also that X is maximal if and only if the spectral sequence degenerates at E_2 and the action of G on $H^*(X, \mathbb{Z}/2\mathbb{Z})$ is trivial. Since we assume maximality for every stratum of our degeneration, we could try to apply this argument to each stratum and then glue them together to relate complex and real geometry of the general fiber; see Section 4.3 for more details.

1.4.3. *Real logarithmic geometry.* From a topological point of view, as remarked in [Bru22], the general real fiber is the union of coverings of the real strata of the special fiber. Naively, one can hope to construct a stratification of the real general fiber from the one of the real special fiber gluing this covering.

Unfortunately, in our general setting, it is unclear to us how to compare this kind of constructions with the complex geometry of the general fiber, since the gluing conditions might be quite complicated. In toric/combinatorial setting, this comparison has been essentially done in [RS23], by carefully codifying the real and complex stratifications in different combinatorial data and then comparing the latter.

To be able to work in our abstract setting, we need to use real and complex logarithmic geometry to simultaneously stratify the complex and the real general fiber; see Sections 3 and 4.

More precisely, there is a natural structure X^{log} and C^{log} of real log-variety, in the sense of [Arg21], on X and C making the morphism $X^{log} \to C^{log}$ a smooth morphism of real log-varieties. Taking the fiber in 0, we get a morphism $X_0^{log} \to 0^{log}$. Then we can take the analytification, in the sense of Kato-Nakayama [KN99a], of this morphism in order to construct a morphism of C^∞-manifolds with corners $X_0^{log}(\mathbb{C}) \to S^1$ endowed with involutions. The main point of logarithmic geometry is that the fiber $X_{0,1}^{log}(\mathbb{C})$ at 1 of this morphism is homeomorphic to the general fiber of the family ([NO10]), while its real part $X_{0,1}^{log}(\mathbb{R})$ is homeomorphic to the real part of a general (positive) fiber ([Par98] or [Arg21] or [Rau22]). Thus $X_{0,1}^{log}(\mathbb{C})$ is the common ambient space we were looking for.

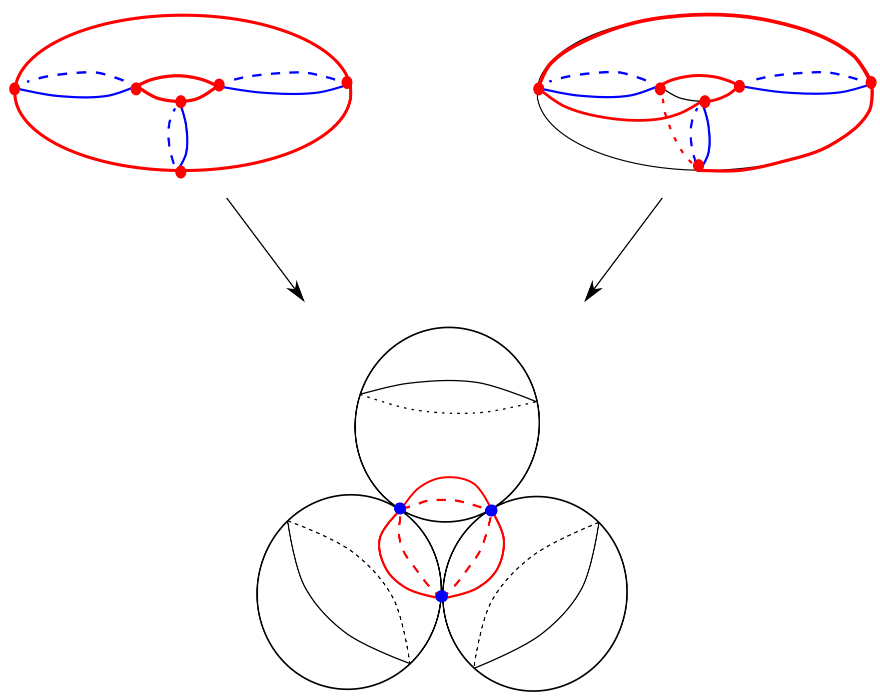

FIGURE 1. $(X_{0,1}^{log}(\mathbb{C}), X_{0,1}^{log}(\mathbb{R})) \to (X_0(\mathbb{C}), X_0(\mathbb{R}))$. On the left a maximal case and on the right a non-maximal one.

It remains to relate this space to the special fiber. The formalism of logarithmic geometry gives a morphism $X_{0,1}^{log}(\mathbb{C}) \to X_0(\mathbb{C})$ compatible with the involutions, which is a nice constructible fibration with fiber at x equal to S^1 to some power n_x, depending only on which stratum X_J^0 the point x lies. This allows to relate the complex geometry of the general fiber to that of the special fiber. To understand the real part, one has just to take the fixed points of the involutions, and thanks to the totally real assumption, one gets a locally constructible cover $\pi : X_{0,1}^{log}(\mathbb{R}) \to X_0(\mathbb{R})$ with fiber at x equal to $\{\pm 1\}$ to some power n_x depending only on which stratum X_J^0 the point x lies; see Section 5.2. The situation is explained in an example in Fig.1, where it is considered a family of real elliptic curves (endowed with two different real structures, respectively on the left and on the right of Fig.1 degenerating to the union of three genus 0 curves.

1.4.4. *Filtred complexes.* In [RS23], in order to prove the main theorem, it is provided an explicit complex which computes combinatorially the real Betti numbers and which is filtered by subcomplexes K_i, in a way such that the resulting graded quotients are isomorphic to the complex computing the tropical homology. The bound is then obtained as a direct consequence of the spectral sequence for a filtered complex. Taking inspiration from the above argument, we now have all the ingredients to prove the desired bound. Once $X_{0,1}^{log}(\mathbb{C})$ and $X_{0,1}^{log}(\mathbb{R})$ are stratified in a compatible way via the stratification of the special fiber, one can run the argument of Section 1.4.2, using equivariant cohomology on each stratum. First, since the real part of the strata have only H^0, the spectral sequence computing the cohomology of a stratified space reduces to a single complex, so that we can compute the cohomology groups of $X_{0,1}^{log}(\mathbb{R})$ via a complex (Section 5.3) whose terms are the cohomology of the strata. Then, since the strata are maximal and have only H^0, the spectral sequence for equivariant cohomology gives a filtration of the real cohomology of each real stratum whose graduated complex is the cohomology of the corresponding complex stratum. Putting these together, we get the filtration that we were looking for and we conclude by using the spectral sequence of a filtered complex. This ends the proof of Theorem 1.3.1(1).

1.4.5. *Hodge numbers.* We finally explain how to prove Theorem 1.3.1(2). By the arguments in [IKMZ19, Pag. 31], the theory of limiting Hodge structures and Theorem 1.3.1's hypothesis, one has that the (p, q)-Hodge numbers of the general fiber can be computed as dimension of the weight $2q$ part in the limiting Hodge structure of $X_{0,1}^{log}(\mathbb{C})$ of degree $p+q$. Hence it is enough to show that the cohomology of the complex $C^\bullet_{q,3,\mathbb{Z}} \otimes \mathbb{Q}$ computes the weight filtration on the limiting Hodge structure.

While this would follows easily from a Hodge theory for the cohomology with compact support of open logarithmic varieties, this theory seems not to be developed. One of the problem is that the open stratum X_J^0 endowed with its natural log structure is not a log-smooth variety, so that one can not naively appeal to some form of logarithmic Poincaré duality.

To avoid this problem, we filter $C^\bullet_{q,3,\mathbb{Z}} \otimes \mathbb{Q}$ by exploiting the Leray spectral sequence for the morphisms $X_{\Delta,1}^{log}(\mathbb{C}) \to X_\Delta^0(\mathbb{C})$, to relate the cohomology of $C^\bullet_{q,3,\mathbb{Z}} \otimes \mathbb{Q}$ to the one of the nearby cycles sheaves on $X_\Delta^0(\mathbb{C})$. At the same time, we compute the weight filtration on $H^n(X_{0,1}^{log}, \mathbb{Q})$ by exploiting the Leray Spectral sequence for the morphism $X_{0,1}^{log}(\mathbb{C}) \to X_0(\mathbb{C})$, also called the vanishing cycles spectral sequence. The main point is that the assumption on the weights allows us to show that the terms appearing in the different spectral sequences involved coincide, since they both compute the same weight part of a mixed Hodge structure. The details of the proof are somehow convoluted and we refer the reader to Section 6.2, for a more precise account.

Remark 1.4.5.1. For related results, which use logarithmic geometry to compute Hodge numbers and to study tropical geometry, see [GB10, RSTZ14, FFR21]

1.5. **Organisation of the paper.** The paper is organised as follows. In Section 2.1, we fix some notation. Section 3 and 4 are devoted to recall and complement some properties of complex and real log-varieties respectively. In Section 5, we prove (1) of Theorem 1.3.1 and in Section 6 we prove (2) of Theorem 1.3.1.

1.6. **Acknowledgement.** The authors are grateful to Erwan Brugallé, Taro Fujisawa, Kris Shaw, Atsushi Shiho for helpful discussions. A thanks to Giuseppe Ancona, Michele Ancona, Olivier Benoist, Erwan Brugallé and Hannah Markwig for useful comments on a preliminary version of this paper. The authors acknowledge the support of Athene Grant and the Centre for Advanced Study (CAS) in Oslo, Norway, which funded and hosted the Young CAS research project Real Structures in Discrete, Algebraic, Symplectic, and Tropical Geometries (REACTIONS) during the 2021-23 academic years.

2. Notation and conventions

2.1. **Stratification.** Let X be a topological space and $\{X_\Delta^0\}$ a collection of locally closed subspaces of X. We write X_Δ for the closure of X_Δ^0. We say that $\{X_\Delta^0\}$ is a stratification of X if

$$X = \coprod X_\Delta^0 \quad \text{and for every } \Delta \text{ one has} \quad X_\Delta = \coprod_{X_{\Delta'}^0 \subseteq X_\Delta} X_{\Delta'}^0.$$

Let $\{X^0_{\Delta'}\}$ another stratification of X. We say that $\{X^0_{\Delta'}\}$ is a refinement of $\{X^0_{\Delta}\}$ if every $X^0_{\Delta'}$ is included is some X^0_{Δ} and for every X^0_{Δ} the collection $\{X^0_{\Delta'}\}_{X^0_{\Delta'}\subseteq X^0_{\Delta}}$ is a stratification of X^0_{Δ}.

2.2. **Degenerations.** Let C be a smooth curve over a field k, a point $0 \in C(k)$ and $f : X \to C$ a projective morphism of smooth varieties. We say that f is semistable around 0 if it is smooth over $C - \{0\}$ and if, étale locally around 0, it is isomorphic to the standard semistable degeneration $\mathrm{Spec}(k[x_1, \dots, x_n, T]/(x_1 \dots x_n - T)) \to \mathrm{Spec}(k[T])$. We say that $f : X \to C$ is strictly semistable around 0 if it is semistable and the irreducible components $\{X_i\}_{i\in I}$ of the fiber X_0 in 0 are smooth. Setting for every subset $J \subseteq I$

$$X_J := \bigcap_{i\in J} X_i \quad \text{and} \quad X^0_J := X_J \setminus \bigcup_{i\notin J} X_i,$$

the collection $\mathfrak{I} := \{X^0_J\}_{J\subseteq I}$ is a stratification of X_0 by smooth algebraic subvarieties. If $k = \mathbb{R}$, we say that f is totally real, if the irreducible components of $X_0(\mathbb{C})$ are real.

2.3. **Local orientations.** Let C be a smooth real curve and $0 \in C(\mathbb{R})$ a real point. The set $C(\mathbb{R})$ is homeomorphic to a disjoint union of open subsets of S^1's. Denote with $\mathcal{H}$ the real connected component of $C(\mathbb{R})$ containing 0. We say that $t \in C(\mathbb{R})$ is *near* 0, if t belongs to $\mathcal{H}$. Let $D \subseteq \mathbb{C}$ be an open disc centered in $(0, 0)$ and endowed with the canonical involution induced by the complex conjugation acting on $\mathbb{C}$.

An *orientation around* $0 \in C$ is a pair (U, ψ) where U is an open neighbourhood of 0 in $C(\mathbb{C})$ stable under complex conjugation and ϕ is an equivariant homeomorphism $\psi : U \to D$ sending 0 to 0. An orientation induces an homeomorphism $\phi_{\mathbb{R}} : U(\mathbb{R}) \xrightarrow{\simeq} (-1, 1)$, where $U(\mathbb{R})$ is the set of points of U fixed by the involution of C. Let $U_+ := \phi^{-1}_{\mathbb{R}}((0, 1))$. A point $t \in C(\mathbb{R})$ near 0 is said to be *positive* with respect to an orientation (U, ψ) if it lies in $(\mathcal{H} \cap U_+) \setminus \{0\}$; otherwise *negative*.

Assume that $X \to C$ is a morphism of real algebraic varieties. A fiber $f^{-1}(t)$, for $t \in C(\mathbb{R})$, is said to be *near* $f^{-1}(0)$ if t is near 0. Moreover, for some fixed orientation (U, ϕ) around 0, a fiber $f^{-1}(t)$ is said to be *positive* if t is near 0 and it is positive with respect to (U, ψ); otherwise *negative*.

3. Log-varieties and analytification

In this section we construct the complex $C^{\bullet}_{p,\mathfrak{Z},A}$ appearing in Theorem 1.3.1 and we state Theorem 1.3.1 in a more precise way; see Theorem 3.4.2. To this end, we need to recall a few notions from logarithmic geometry (Section 3.1), and from their associated analytification (Section 3.2). Then we study the geometry of certain stratifications of these analytifications (Section 3.3) and we finally construct the complex $C^{\bullet}_{p,\mathfrak{Z},A}$ in Section 3.4.

3.1. Logarithmic varieties.

3.1.1. *Definitions and examples.* We start recalling a few notions from logarithmic geometry. For more details see for example [Ill02] and [Kat89].

Let k be a field and X a k-variety. We consider $\mathcal{O}_X$ as a sheaf of (abelian) monoids with the multiplication operation.

Definition 3.1.1.1.
(1) A pre-log structure on X is a morphism of sheaf of (abelian) monoids $\alpha_X : M \to \mathcal{O}_X$.
(2) A pre-log structure $\alpha_X : M \to \mathcal{O}_X$ is said a log structure if the natural morphism $\alpha_{|\alpha^{-1}(\mathcal{O}^*_X)} : \alpha^{-1}(\mathcal{O}^*_X) \to \mathcal{O}^*_X$ is an isomorphism.
(3) Given a pre-log structure $\alpha_X : M \to \mathcal{O}_X$, there is a canonical associated log structure $\alpha'_X : M \to \mathcal{O}_X$; see [Kat89, Section 1.3].
(4) A log-variety is a pair (X, M) where X is a k-variety and M a log structure on X.

All the logarithmic structures in this paper will be assumed to be fine and saturated ([Ill02, 1.3]) and all the operations between log-varieties will be done in the category of fine and saturated log-varieties. Given a log-variety (X, M), we consider $\mathcal{O}^*_X$ as a submonoid of M via α. We write M^{gr} for the groupification of the sheaf of monoids M and $\overline{M}$ (resp. $\overline{M}^{gr}$) for the quotient $M/\mathcal{O}^*_X$ (resp. $\overline{(M/\mathcal{O}^*_X)}^{gr}$). There is a natural notion of morphism between logarithmic varieties. Given a log-variety (X, M) and a morphism of k-varieties $f : Y \to X$, there is an induced log structure f^*M on Y ([Kat89, Section 1.4]) and a morphism of log-varieties $f : (Y, f^*M) \to (X, M)$. Here some examples that are fundamental for our purposes and that will be treated further in Examples 3.2.1.1, 4.1.1.1.

Example 3.1.1.2. In the following examples, if a morphism of monoids $\mathbb{N} \to \mathbb{N}^r$ is given, it is the diagonal map which sends 1 to $\sum_{i=1}^{r} e_i$, where e_i denotes the n-uple with the i-th component equal to 1 and the remaining to 0.

(1) The trivial log structure on X is given by $(X, \mathcal{O}_X^*)$ and α_X is the natural inclusion $\mathcal{O}_X^* \to \mathcal{O}_X$. In this way any variety can be consider as a log-variety with the trivial log structure. If (X, M) is a log-variety, there is a canonical morphism $(X, M) \to X$.

(2) Let $X = \operatorname{Spec}(k)$ and let $\mathbb{N} \to k$ be the map sending 1 to 0. The associated log structure $\mathbb{N} \times k^* \to k$ sends $(1, \lambda)$ to 0 and $(0, \lambda)$ to λ. It is called the standard log structure on the point and we write $\operatorname{Spec}(k)^{\log}$ for the associated log-scheme.

(3) Let C be $\mathbb{A}^1 = \operatorname{Spec}(k[T])$ and let $\mathbb{N} \to k[T]$ be the map sending 1 to T. The associated log structure $\mathbb{N} \times k^* \to k[T]$ sends (n, f) to fT^n. It is called the log structure on $\mathbb{A}^1$ associated to the divisor 0. We write C^{log} for the associated log-scheme. The restriction of such log structure to $0 \in \mathbb{A}^1 = \operatorname{Spec}(k[T])$, induces the standard log structure on the point.

(4) Let $X = \operatorname{Spec}(k[x_1 \dots x_n, T]/(x_1 \dots x_n - T))$ and $\mathbb{N}^n \to k[x_1 \dots x_n, T]/(x_1 \dots x_n - T)$ be the pre-log structure sending e_i to x_i. The associated log structure $\mathbb{N}^n \times k^* \to k[x_1 \dots x_n, T]/(x_1 \dots x_n - T)$ sends (e_i, f) to fx_i. We write X^{log} for the associated log-variety. If $C = \mathbb{A}^1$ is endowed with the log structure in Example 3.1.1.2(3), the natural morphism $X \to C$ extends to a morphism of log-varieties $X^{log} \to C^{log}$ described by the commutative diagram

$$\begin{tikzcd} \mathbb{N} \arrow[r, "1 \mapsto T"] \arrow[d, "1 \mapsto \sum e_i"] & k[T] \arrow[d] \\ \mathbb{N}^n \arrow[r, "e_i \mapsto x_i"] & k[x_1 \dots x_n, T]/(x_1 \dots x_n - T). \end{tikzcd}$$

(5) Retain the notation of Example 3.1.1.2(4) and take the fiber $X_0^{log} \to 0^{log}$ at 0 of the morphism $X^{log} \to C^{log}$. Then the situation is described by the commutative diagram

$$\begin{tikzcd} \mathbb{N} \arrow[r, "1 \mapsto 0"] \arrow[d, "1 \mapsto \sum e_i"] & k \arrow[d] \\ \mathbb{N}^n \arrow[r, "e_i \mapsto x_i"] & k[x_1 \dots x_n]/(x_1 \dots x_n). \end{tikzcd}$$

(6) Retain the notation of Example 3.1.1.2(5), set $J = \{1, \dots, r\}$ and endow the open strata

$$X_J^0 := \operatorname{Spec}(k[x_{r+1}^{\pm 1} \dots x_n^{\pm 1}]) \subseteq X_0,$$

with the log structure induced by the pre-log structure $\mathbb{N}^r \to k[x_{r+1}^{\pm 1} \dots x_n^{\pm 1}]$ sending e_i to 0. We write $X_J^{0,log}$ for the corresponding log-variety. The natural morphism $X_J^{0,log} \to 0^{log}$ is described by the commutative diagram

$$\begin{tikzcd} \mathbb{N} \arrow[r, "1 \mapsto 0"] \arrow[d, "1 \mapsto \sum e_i"] & k \arrow[d] \\ \mathbb{N}^r \arrow[r, "e_i \mapsto 0"] & k[x_{r+1}^{\pm 1} \dots x_n^{\pm 1}]. \end{tikzcd}$$

3.1.2. *The case of a semistable family.* More generally, let C be a smooth k-curve and $0 \in C(k)$. Let $X \to C$ be a projective family, smooth over $U_C := C \setminus \{0\}$ and strictly semistable around 0.

The natural inclusion $\mathcal{O}_C \cap j_*\mathcal{O}_{U_C}^* \to \mathcal{O}_C$ induces a log structure on C, called the log structure associated to the divisor 0 in C. We write $C^{\log}$ for the corresponding log-variety. Locally around 0, in the étale topology, the logarithmic structure is given by the one in Example 3.1.1.2(3); see [Kat89, Example 1.5].

Let $U_X := X \setminus X_0 \subseteq X$. The natural inclusion $\mathcal{O}_X \cap j_*\mathcal{O}_{U_X}^* \to \mathcal{O}_X$ is a pre-log structure on X, which induces a log-variety X^{log} and a morphism of log-varieties $X^{\log} \to C^{\log}$, which, locally around 0, is isomorphic to the one given in Example 3.1.1.2(4). The restriction of such log structure to X_0 induces a canonical log structure on X_0, with associated log-variety $X_0^{\log}$, and a morphism $X_0^{\log} \to \operatorname{Spec}(k)^{\log}$ (see [Kat89, Example 3.7 (2)]), which is locally isomorphic to the one given in Example 3.1.1.2(5).

Endowing the strata $X_J^0 \subseteq X_0$ with the induced log structure, we get a closed immersion of log-schemes $X_J^{0,log} \subseteq X_0^{log}$, which is locally isomorphic to the one in Example 3.1.1.2(6). The situation is summarized in the following commutative diagram:

$$\begin{tikzcd}
X_J^{0,log} \arrow[rrr, "\pi"] \arrow[d] & & & X_J^0 \arrow[d] \\
X_0^{log} \arrow[r] \arrow[d, "f_0^{log}"] \arrow[rrr, bend left, "\pi"] & X^{log} \arrow[r, "\pi"] \arrow[d, "f^{log}"] & X \arrow[d, "f"] & X_0 \arrow[l] \arrow[d, "f_0"] \\
\mathrm{Spec}(k)^{log} \arrow[r, "0"] \arrow[rrr, bend right, "\pi"'] & C^{log} \arrow[r, "\pi"] & C & \mathrm{Spec}(k) \arrow[l, "0"']
\end{tikzcd}$$

3.2. **Analytification.** Assume now that $k = \mathbb{C}$ and (X, M) is a log $\mathbb{C}$-variety. We recall how to associate to (X, M) a topological space, the Kato-Nakayama space of (X, M), which is a topological incarnation of (X, M) extending the usual analytification for $\mathbb{C}$-varieties. We refer the reader to [KN99b] for the original construction and to [Ill02, Section 5.5] for a nice introduction.

3.2.1. *Definitions and examples.* More precisely, there is a natural functorial way to associate to (X, M) a C^∞-manifold with corner $(X, M)^{an}$, called the Kato-Nakayama space of (X, M). Concretely, consider the pair $(\mathrm{Spec}(\mathbb{C}), \pi)$ where $\pi : \mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C}$ is the polar coordinates defined by $\pi(x, y) = xy$. Then, by definition,

$$(X, M)^{an} := \mathrm{Hom}((\mathrm{Spec}(\mathbb{C}), \pi), (X, M))$$

endowed with a natural topology. We will often write $(X, M)(\mathbb{C})$ for $(X, M)^{an}$.

If M is the trivial log structure, then $(X, M)^{an} = X(\mathbb{C})$ is the usual analytification. In general, for every log-variety (X, M), the natural morphism $(X, M) \to X$ induces a proper morphism $(X, M)^{an} \to X^{an}$, whose fiber over a point $x \in X^{an}$ is $(S^1)^{\mathrm{rank}(\overline{M}_x^{gp})}$; see the notation introduced in Section 3.1.1. Here some examples which are relevant for this paper.

Example 3.2.1.1.
(1) As already mentioned, if X has the trivial log structure then $X^{an} = X(\mathbb{C})$.
(2) In the notation of Example 3.1.1.2(2), $(Spec(\mathbb{C})^{\log})^{an} = S^1$, since the only morphisms $h : \mathbb{N} \to \mathbb{R}_{\geq 0} \times S^1$ making the diagram

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "h"] \arrow[d, "1 \mapsto 0"] & \mathbb{R}_{\geq 0} \times S^1 \arrow[d, "(x,y) \mapsto xy"] \\
\mathbb{C} \arrow[r, "id"] & \mathbb{C}
\end{tikzcd}$$

commutative are those sending 1 to $(0, y)$ for $y \in S^1$.
(3) In the notation of Example 3.1.1.2(3), then $C^{log,an} \simeq \mathbb{R}_{\geq 0} \times S^1$ is the real blow-up of $\mathbb{C}$ at 0, as one readily sees by looking at the set of commutative diagrams

$$\begin{tikzcd}
\mathbb{N} \arrow[r] \arrow[d, "1 \mapsto T"] & \mathbb{R}_{\geq 0} \times S^1 \arrow[d, "(x,y) \mapsto xy"] \\
\mathbb{C}[T] \arrow[r, two heads, "T \mapsto a_i"] & \mathbb{C}.
\end{tikzcd}$$

Moreover, the natural map $C^{log,an} \simeq \mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C} \simeq C^{an}$ is the usual real-blow-up map (the polar coordinates).
(4) Retain the notation of Example 3.1.1.2(6). Then $X_J^{0,\log}(\mathbb{C}) \simeq (S^1)^r \times (\mathbb{C}^*)^{n-r}$, as one readily sees by looking at the set of commutative diagrams

$$\begin{tikzcd}
\mathbb{N}^r \arrow[r, two heads, "e_i \mapsto (0, y_i)"] \arrow[d, "e_i \mapsto 0"] & \mathbb{R}_{\geq 0} \times S^1 \arrow[d, "(x,y) \mapsto xy"] \\
\mathbb{C}[x_{r+1}^{\pm 1} \dots x_n^{\pm 1}] \arrow[r, two heads, "x_i \mapsto a_i \neq 0"] & \mathbb{C}.
\end{tikzcd}$$

and the map $X_J^{0,\log}(\mathbb{C}) \to X_J^0(\mathbb{C}) \simeq (\mathbb{C}^*)^{n-r}$ is the natural projection.

The natural map $X_J^{0,\log} \to \mathrm{Spec}(\mathbb{C})^{\log}$ induces a map $X_J^{0,\log}(\mathbb{C}) \to S^1$, which sends

$$((y_1 \dots y_r), (a_{r+1} \dots a_n)) \quad \text{to} \quad \prod_i y_i.$$

Indeed, the point $((y_1 \dots y_r), (a_{r+1} \dots a_n))$ corresponds to the commutative diagram

$$\begin{tikzcd} \mathbb{N}^r \arrow[r, "e_i \mapsto (0,y_i)"] \arrow[d, "e_i \mapsto 0"] & \mathbb{R}_{\geq 0} \times S^1 \arrow[d, "(x,y)\mapsto xy"] \\ \mathbb{C}[x_{r+1}^{\pm 1} \dots x_n^{\pm n}] \arrow[r, two heads, "x_i \mapsto a_i"] & \mathbb{C}. \end{tikzcd}$$

and its image in S^1 it is obtained by precomposing it with the diagram,

$$\begin{tikzcd} \mathbb{N} \arrow[r, "1 \mapsto \sum e_i"] \arrow[d, "1 \mapsto 0"] & \mathbb{N}^r \arrow[d, "e_i \mapsto 0"] \\ \mathbb{C} \arrow[r] & \mathbb{C}[x_{r+1}^{\pm 1} \dots x_n^{\pm n}] \end{tikzcd}$$

in which the bottom horizontal map is the canonical inclusion. Finally, one obtains the diagram

$$\begin{tikzcd} \mathbb{N} \arrow[r, "e_i \mapsto (0, \prod_i y_i)"] \arrow[d] & \mathbb{R}_{\geq 0} \times S^1 \arrow[d, "(x,y)\mapsto xy"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C}, \end{tikzcd}$$

which corresponds to the point $\prod y_i \in S^1$. In particular, the fiber $X_{J,1}^{0,\log}(\mathbb{C})$ at 1 of this morphism is the subset of $X_J^{0,\log}(\mathbb{C})$, in which the product of the first r coordinates is 1.

3.2.2. *The case of a semistable degeneration.* As in Section 3.1.2, assume from now that $X \to C$ is a real strictly semistable degeneration endowed with the logarithmic structure. Choose an orientation (U, ψ) around 0; see Section 2.3. Then, by Example 3.2.1.1(3), the orientation induces an isomorphism $\pi^{-1}(U) \simeq [0,1) \times S^1$, where

$$\pi : C^{log,an} \to C^{an}$$

is the natural map. Hence $C^{log,an}$ can be identified with the real oriented blow-up of C^{an} in 0; see e.g. [NO10, (1.2.3)].

Then, we can analytify the commutative diagram

$$\begin{tikzcd} X_J^{0,log} \arrow[r, "\pi"] \arrow[d] & X_J^0 \arrow[d] \\ X_0^{log} \arrow[r, "\pi"] \arrow[d] & X_0 \arrow[d] \\ 0^{log} \arrow[r] & 0 \end{tikzcd}$$

to get morphisms $X_J^{0,log}(\mathbb{C}) \to X_0^{log}(\mathbb{C}) \to 0^{log,an} \simeq S^1$ with fibers at 1 given by $X_{J,1}^{log}(\mathbb{C}) \to X_{0,1}^{log}(\mathbb{C})$ and a commutative cartesian diagram

$$\begin{tikzcd} X_{J,1}^{0,log}(\mathbb{C}) \arrow[r] \arrow[d] & X_J^{0,log}(\mathbb{C}) \arrow[r, "\pi"] \arrow[d] & X_J^0(\mathbb{C}) \arrow[d] \\ X_{0,1}^{log}(\mathbb{C}) \arrow[r] \arrow[d] & X_0^{log}(\mathbb{C}) \arrow[r, "\pi"] \arrow[d] & X_0(\mathbb{C}) \\ \{1\} \arrow[r] & 0^{\log}(\mathbb{C}) = S^1. & \end{tikzcd}$$

The main result that we are interested in and which motivates the use of logarithmic geometry, is the following.

Theorem 3.2.2.1. [NO10, Theorem 0.3] *The topological space $X_{0,1}^{log}(\mathbb{C})$ is homeomorphic to every smooth fiber of the morphism $X \to C$.*

Hence, in order to study the Betti numbers of the complex general fiber of $X \to C$, we can study the topology of $X_{0,1}^{log}(\mathbb{C})$ via the morphism $X_{0,1}^{log}(\mathbb{C}) \to X_0(\mathbb{C})$.

3.3. **Stratifications.** Now, we go further and study $X_{0,1}^{log}(\mathbb{C})$, by stratifying it in a way which is compatible with the logarithmic structure on $X_0(\mathbb{C})$. Retain the notation of Sections 3.2.2 and 2.2.

Let $\mathfrak{Z} := \{X_\Delta^0\}$ be a refinement of $\mathfrak{I}$ made by smooth algebraic varieties. Let I_Δ be the subset $I_\Delta \subseteq I$ such that $X_\Delta^0 \subseteq X_{I_\Delta}^0$ and set $|\Delta| := |I_\Delta|$. We endow X_Δ^0 with the structure of a log-variety $X_\Delta^{0,log}$ by restricting the log structure of X_0 to X_Δ^0. The most important properties of $X_\Delta^{0,log}(\mathbb{C})$ and $X_{\Delta,1}^{0,log}(\mathbb{C})$, which follow directly from the local description given in Example 3.2.1.1(4), are summarized in the following remark.

Remark 3.3.1.
(1) The morphism $X_\Delta^{0,log}(\mathbb{C}) \to X_\Delta^0(\mathbb{C})$ is a locally trivial fibration with fibers $(S^1)^{|\Delta|}$. In particular $X_\Delta^{0,log}(\mathbb{C})$ is a smooth C^∞-manifold of dimension $2\dim(X_\Delta^0) + |\Delta|$.
(2) The morphism $X_{\Delta,1}^0(\mathbb{C}) \to X_\Delta^0(\mathbb{C})$ is a proper locally trivial fibration with fiber $(S^1)^{|\Delta|-1}$. In particular, $X_\Delta^{0,log}(\mathbb{C})$ is a smooth C^∞-manifold of dimension $2\dim(X_\Delta^0) + |\Delta| - 1$.

Since $\{X_\Delta^0(\mathbb{C})\}$ is a stratification of X_0 and $\overline{X_\Delta^{0,\log}(\mathbb{C})} = X_\Delta^{\log}(\mathbb{C}) \subseteq X_0^{log}(\mathbb{C})$[1], we see that $\{X_\Delta^{0,log}(\mathbb{C})\}$ is a stratification of $X_0^{log}(\mathbb{C})$. Since $X_{0,1}^{log}(\mathbb{C}) \subseteq X_0^{log}(\mathbb{C})$ is a closed subset, also $\{X_{\Delta,1}^{0,log}(\mathbb{C})\}$ is a stratification of $X_{0,1}^{log}(\mathbb{C})$.

3.4. **Construction of the complex and statement of the main theorem.** Retain the notation of Section 3.3 and let A be a coefficient ring. There is a natural spectral sequence (see e.g. [Pet16] and [Pet17, (3) Pag. 2527])

$$ {}_A^{\mathbb{C}}E_1^{a,b} := \bigoplus_{\dim(X_\Delta^0)=a} H_c^{a+b}(X_{\Delta,1}^{0,log}(\mathbb{C}), A) \Rightarrow H_c^{a+b}(X_{0,1}^{log}(\mathbb{C}), A)\big(\simeq H_c^{a+b}(X_t(\mathbb{C}), A)\big), \tag{3.4.1} $$

where $t \in C(\mathbb{C})$ is any member different from 0 and the last isomorphism follows from Theorem 3.2.2.1.

Set

$$ C_{q,\mathfrak{Z},A}^\bullet := {}_A^{\mathbb{C}}E_1^{\bullet,q} : $$

$$ \bigoplus_{\dim(X_\Delta^0)=p-1} H_c^{p+q-1}(X_{\Delta,1}^{0,log}(\mathbb{C}), A) \to \bigoplus_{\dim(X_\Delta^0)=p} H_c^{p+q}(X_{\Delta,1}^{0,log}(\mathbb{C}), A) \to \bigoplus_{\dim(X_\Delta^0)=p+1} H_c^{p+q+1}(X_{\Delta,1}^{0,log}(\mathbb{C}), A). $$

Now Theorem 1.3.1 can be stated more precisely as follows.

Theorem 3.4.2.
(1) With the notation of Theorem 1.3.1, assume that
(a) $H^i(X_\Delta^0(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = 0$ for all $i \geq 1$ and $X_\Delta^0 \in \mathfrak{Z}$;
(b) X_Δ^0 is a maximal variety, for all $X_\Delta^0 \in \mathfrak{Z}$.
Then, for every $t \in C(\mathbb{R})$ near 0 one has:

$$ b_p(X_t(\mathbb{R})), \mathbb{Z}/2\mathbb{Z}) \leq \sum_{0 \leq q \leq n} dim(H^p(C_{q,\mathfrak{Z},\mathbb{Z}/2\mathbb{Z}}^\bullet). $$

(2) If in addition to (a) and (b), we assume that
(c) the mixed Hodge structure on $H^i(X_\Delta^0, \mathbb{Q})$ is pure of type (i,i) and $H^i(X_\Delta^0, \mathbb{Z})$ is torsion free,
Then, for every $t \in C(\mathbb{R})$ near 0 one has:

$$ (i) \quad dim(H^q(C_{p,\mathfrak{Z},\mathbb{Z}}^\bullet \otimes \mathbb{Q})) = h^{p,q}(X_t) \quad and \quad (ii) \quad C_{p,\mathfrak{Z},\mathbb{Z}}^\bullet \otimes \mathbb{Z}/2\mathbb{Z} \simeq C_{p,\mathfrak{Z},\mathbb{Z}/2\mathbb{Z}}^\bullet. $$

[1]This can be checked locally, hence we can assume that $X_0^{\log} = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n]/(x_1 \dots x_n))$ endowed with the logarithmic structure of Example 3.1.1.2(5). Since in this case $X_0^{\log}(\mathbb{C})$ is closed inside $\mathbb{A}^{n,log}(\mathbb{C}) \simeq (\mathbb{R}_{\geq 0} \times S^1)^n$, we can assume that X_0^{log} is $\mathbb{A}^n = \prod \mathbb{A}^1$ with the product log structure of Example 3.1.1.2(3). In this case the result follows from a computation using that the map $(\mathbb{R}_{\geq 0} \times S^1)^n \simeq \mathbb{A}^{n,\log}(\mathbb{C}) \to \mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ is the polar coordinates componentwise.

4. Real log-varieties and equivariant cohomology

In this section we recall some facts from the theory of real logarithmic varieties. The theory appears explicitly in the literature in [Arg21, Sections 5-7] and in [AB21], and it is implicit in [Rau22] and [Par98]. We borrow part of the presentation from [Arg21, Sections 5-7], where the author works in a slightly more general setting and we complement Argüz' results adding what is needed for our purposes. In Section 4.1, we recall the definitions, examples and we study the case of totally real semistable degenerations. In Section 4.2 we study stratifications of these degenerations. In Section 4.3 we recall a few facts around equivariant cohomology.

4.1. **Real logarithmic structure.**

4.1.1. *Definitions and examples.* Assume now that $k = \mathbb{R}$. To give a log-variety over $\mathbb{R}$ is equivalent to give a log-variety (X, M) over $\mathbb{C}$ endowed with a pair (σ, σ_M) where $\sigma : X(\mathbb{C}) \to X(\mathbb{C})$ is an anti-holomorphic involution and $\sigma_M : M \to M$ is an involution of monoids making the diagram

$$\begin{array}{ccc} M & \xrightarrow{\sigma_M} & M \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \xrightarrow{\sigma} & \mathcal{O}_X \end{array}$$

commutative, where, by abuse of notation, we denote by σ also the involution on $\mathcal{O}_X$ induced by that on $X(\mathbb{C})$.

We endow $\mathbb{R}_{\geq 0} \times S^1$ with the involution τ given by $\tau(x, r) = (x, \overline{r})$, so that the polar coordinates morphism $\mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C}$ is equivariant once $\mathbb{C}$ is endowed with the standard conjugation. Then τ and σ induce a natural involution on $(X, M)(\mathbb{C}) := \mathrm{Hom}((\mathrm{Spec}(\mathbb{C}), \pi), (X, M))$ and the real locus $(X, M)(\mathbb{R}) \subseteq (X, M)(\mathbb{C})$ of (X, M) is, by definition, the fixed locus of this involution endowed with its subspace topology.

If $(X, M) \to (Y, N)$ is a morphism of real log-varieties, then the induced morphism $(X, M)(\mathbb{C}) \to (Y, N)(\mathbb{C})$ is equivariant and hence it induces a morphism $(X, M)(\mathbb{R}) \to (Y, N)(\mathbb{R})$. In particular, there is a natural morphism $(X, M)(\mathbb{R}) \to X(\mathbb{R})$. Now, we give the examples that are relevant for our paper.

Example 4.1.1.1.

(1) If X is endowed with the trivial log structure, then $(X, M)(\mathbb{R}) \simeq X(\mathbb{R})$ is just the set of real points of X.

(2) Let $X = \mathrm{Spec}(\mathbb{R})$ endowed with the canonical log structure. Then the induced action on $(X(\mathbb{C}), \mathbb{N})$ is trivial on both $X(\mathbb{C}) = \{\mathrm{pt}\}$ and $\mathbb{N}$. This induces on the analytification $X^{\log}(\mathbb{C}) \simeq S^1$ the standard conjugation, so that $X^{\log}(\mathbb{R}) = \{+1, -1\}$.

(3) Assume that $C = \mathrm{Spec}(\mathbb{R}[T])$, endowed with the log structure from Example 3.1.1.2(3). Then, the involution on $(C(\mathbb{C}), \mathbb{N})$ is the standard complex conjugation on $C(\mathbb{C}) \simeq \mathbb{C}$ and the identity on $\mathbb{N}$. This induces on the analytification $C^{\log}(\mathbb{C}) \simeq \mathbb{R}_{\geq 0} \times S^1$ (see Example 3.2.1.1(3)) the identity of the first factor and the standard complex conjugation on the second one, so that $C^{log}(\mathbb{R}) \simeq \mathbb{R}_{\geq 0} \times \{\pm 1\}$. The natural map $C^{log}(\mathbb{R}) \simeq \mathbb{R}_{\geq 0} \times \{\pm 1\} \to C(\mathbb{R}) \simeq \mathbb{R}$ sends $(x, 1)$ to x and $(x, -1)$ to $-x$.

(4) Assume that $X_J^0 = \mathrm{Spec}(\mathbb{R}[x_{r+1}^{\pm 1} \dots x_n^{\pm 1}])$, endowed with the log structure $\mathbb{N}^r \to \mathbb{R}[x_{r+1}^{\pm 1} \dots x_n^{\pm 1}]$ from Example 3.1.1.2(6) sending all non zero elements to 0. Then the involution on $(X_J^0(\mathbb{C}), \mathbb{N}^r)$ is the standard complex conjugation on $X_J^0(\mathbb{C}) \simeq (\mathbb{C}^*)^{n-r}$ and the trivial action on $\mathbb{N}^r$. Hence the induced action on $X_J^{0,\log}(\mathbb{C}) \simeq (S^1)^r \times (\mathbb{C}^*)^{n-r}$ (see Example 3.2.1.1(4)) is the standard complex conjugation acting component-wise and without any permutation of the factors, so that $X_J^{0,\log}(\mathbb{R}) \simeq \{\pm 1\}^r \times (\mathbb{R}^*)^{n-r}$ and the map $X_J^{0,\log}(\mathbb{R}) \simeq \{\pm 1\}^r \times (\mathbb{R}^*)^{n-r} \to X(\mathbb{R}) \simeq (\mathbb{R}^*)^{n-r}$ is the natural projection. The natural map $X_J^{0,\log}(\mathbb{R}) \simeq \{\pm 1\}^r \times (\mathbb{R}^*)^{n-r} \to \mathrm{Spec}(\mathbb{R})^{\log}(\mathbb{R}) \simeq \{\pm 1\}$ sends $((y_1 \dots y_r), (a_{r+1} \dots a_r))$ to $\prod_i y_i$, as follows from the computation done in Example 3.2.1.1(4). In particular, the fiber $X_{J,1}^{0,\log}(\mathbb{R})$ at 1 of this morphism identifies with the subset of $X_J^{0,\log}(\mathbb{R}) \simeq \{\pm 1\}^r \times (\mathbb{R}^*)^{n-r}$ in which the product of the first r components is 1. Hence $X_{J,1}^{0,\log}(\mathbb{R}) \to X_J^0(\mathbb{R})$ is a topological cover of degree $2^{|J|-1}$; see also [Arg21, Proposition 8.5] and [Rau22, Theorem 1.3].

In all the examples of Example 4.1.1.1, the action of the involution on the sheaf of monoids is trivial. Even if this is the only case we will need (since the degenerations we are interested in are totally real), for completeness, now we give an example in which the action comes from a real semistable degeneration which in not totally real and, therefore, it is non-trivial. The reader only interested in Theorem 1.3.1 can skip the end of this section.

Example 4.1.1.2. Let X_J^0 be $\mathrm{Spec}(\mathbb{C}[x_{r+1}^{\pm 1} \dots x_n^{\pm 1}])$ and endow it with the logarithmic structure associated to the morphism $\mathbb{N}^r \to \mathrm{Spec}(\mathbb{C}[x_{r+1}^{\pm 1} \dots x_n^{\pm 1}])$ sending all the non zero element to 0. We put the standard conjugation action on $X_J^0(\mathbb{C})$, but now we consider $\mathbb{N}^r$ endowed with the involution switching e_1 and e_2 and acting trivially on the other e_i.

This corresponds to the restriction to some geometrically irreducible strata of the log structure coming from a degeneration whose special fiber has the form $\mathrm{Spec}(\mathbb{R}[x_1 \dots x_n]/((x_1^2+x_2^2)x_3 \dots x_n)$, which is not totally real.

Then, the induced involution on $X_J^0(\mathbb{C}) \simeq (S^1)^r \times (\mathbb{C}^*)^{n-r}$ sends $((y_1, y_2, y_3 \dots y_r), (a_{r+1} \dots a_n))$ to

$$((\overline{y_2}, \overline{y_1}, \overline{y_3} \dots \overline{y_r}), (\overline{a_{r+1}} \dots \overline{a_n})),$$

so that $X_J^{0,\log}(\mathbb{R}) \subseteq X_J^{0,\log}(\mathbb{C})$ identifies with the subset made by the elements of the form

$$((y_1, \overline{y_1}, \pm 1, \dots, \pm 1), (a_{r+1} \dots a_n))$$

with $y_1 \in S^1$ and $a_i \in \mathbb{R}^*$. Hence $X_J^{0,\log}(\mathbb{R})$ identifies with $S^1 \times \{\pm 1\}^{r-2} \times (\mathbb{R}^*)^{n-r}$.

The natural map $S^1 \times \{\pm 1\}^{r-2} \times (\mathbb{R}^*)^{n-r} \simeq X_J^{0,log}(\mathbb{R}) \to X_J^0(\mathbb{R}) \simeq (\mathbb{R}^*)^{n-r}$ is the natural projection, which is no longer a topological cover of degree 2^r but a product of a 2^{r-2} topological cover and an S^1-bundle over $X_J^0(\mathbb{R})$. The fiber at 1 of the other natural map $S^1 \times \{\pm 1\}^{r-2} \times (\mathbb{R}^*)^{n-r} \simeq X_J^{0,log}(\mathbb{R}) \to \mathrm{Spec}(\mathbb{R})^{\log} \simeq \{\pm 1\}$ is given (since $y_1\overline{y_1} = 1$) by the subset of $X_J^{0,log}(\mathbb{R})$ made by elements $((y_1, \overline{y_1}, y_3 \dots y_r), (a_{r+1} \dots a_n))$ for which $\prod_{i \geq 3}(y_i) = 1$, hence it is a product of an S^1-bundle and a topological cover of degree 2^{r-3} over $X_J^0(\mathbb{R})$.

4.1.2. *The case of a strictly semistable degeneration.* Assume from now on that $X \to C$ is a totally real strictly semistable degeneration and endow it with the logarithmic structures as in Section 3.1.2. Then we have a morphisms of real log-varieties $X^{log} \to C^{log}$ and $X_0^{log} \to 0^{log}$. Locally around 0, the log structure is isomorphic to the one given in Example 4.1.1.1. Fix an orientation (U, ϕ) around 0; see Section 2.3.

Recall the commutative diagram from Section 3.2.2

$$\begin{tikzcd}
X_{J,1}^{0,log}(\mathbb{C}) \arrow[r] \arrow[d] & X_J^{0,log}(\mathbb{C}) \arrow[r, "\pi"] \arrow[d] & X_J^0(\mathbb{C}) \arrow[d] \\
X_{0,1}^{log}(\mathbb{C}) \arrow[r] \arrow[d] & X_0^{log}(\mathbb{C}) \arrow[r, "\pi"] \arrow[d] & X_0(\mathbb{C}) \\
\{1\} \arrow[r] & 0^{\log}(\mathbb{C}) = S^1. &
\end{tikzcd}$$

The morphisms on the right part of the diagram are compatible with the involutions acting on the various topological spaces appearing. Hence the involutions acting on $X_J^{0,log}(\mathbb{C})$ and $X_0^{log}(\mathbb{C})$ restrict to involutions on $X_{J,1}^{0,log}(\mathbb{C})$ and $X_{0,1}^{log}(\mathbb{C})$ respectively, making the whole diagram equivariant.

Taking fixed points of the involutions, we get a commutative diagram

$$\begin{tikzcd}
X_{J,1}^{0,log}(\mathbb{R}) \arrow[r] \arrow[d] & X_J^{0,log}(\mathbb{R}) \arrow[r, "\pi_{\mathbb{R}}"] \arrow[d] & X_J^0(\mathbb{R}) \arrow[d] \\
X_{0,1}^{log}(\mathbb{R}) \arrow[r] \arrow[d] & X_0^{log}(\mathbb{R}) \arrow[r, "\pi_{\mathbb{R}}"] \arrow[d] & X_0(\mathbb{R}) \\
\{1\} \arrow[r] & 0^{\log}(\mathbb{R}) = \{\pm 1\}, &
\end{tikzcd}$$

where $X_{J,1}^{0,log}(\mathbb{R})$ and $X_{0,1}^{log}(\mathbb{R})$ are both the fiber at 1 of the morphisms $X_J^{0,log}(\mathbb{R}) \to X_0^{log}(\mathbb{R}) \to \{\pm 1\}$ and the fixed points of the involutions acting respectively on $X_{J,1}^{0,log}(\mathbb{C})$ and $X_1^{0,log}(\mathbb{C})$. The main result that we are interested in and which motivates the use of real logarithmic geometry, is the real analogue of Theorem 3.2.2.1, which follows from the results in [Par98] or in [Arg21] or in [Rau22].

Theorem 4.1.2.1 ([Par98], [Arg21], [Rau22])**.** *The topological space $X_{0,1}^{log}(\mathbb{R})$ is homeomorphic to every positive real smooth fiber near 0 of the morphism $X \to C$.*

Proof. This is essentially the content of [Arg21, Proposition 7.4] or the one of [Rau22, Theorem 1.1], which say [2] that $X^{log}(\mathbb{R}) \to C^{log}(\mathbb{R})$ is a locally trivial fibration. Therefore all smooth real fibers in the same connected component of $C^{log}(\mathbb{R})$ are homeomorphic. Since the orientation induces an equivariant isomorphism

$$\pi^{-1}(U) \simeq [0,1) \times S^1 \quad \text{so that} \quad \pi_{\mathbb{R}}^{-1}(U(\mathbb{R})) \simeq [0,1) \times \{\pm 1\},$$

and the fibers of $X^{log}(\mathbb{R}) \to C^{log}(\mathbb{R})$ are homeomorphic to the ones of $X(\mathbb{R}) \to C(\mathbb{R})$ outside 0, the positive fibers of $X(\mathbb{R}) \to C(\mathbb{R})$ near 0 are all homeomorphic to $X_{0,1}^{log}(\mathbb{R})$. □

Hence, in order to study the Betti numbers of a positive smooth real general fiber of $X(\mathbb{R}) \to C(\mathbb{R})$, we can study the topology of $X_{0,1}^{log}(\mathbb{R})$ via the morphism $X_{0,1}^{log}(\mathbb{R}) \to X_0(\mathbb{R})$.

4.2. **Real stratifications.** Let $\mathfrak{Z} := \{X_\Delta^0\}_{\Delta \in P}$ be a refinement of $\mathfrak{I}$ made by smooth real algebraic varieties and let us retain the notation of Section 3.3. We endow X_Δ^0 with the structure of a real log-variety $X_\Delta^{0,log}$ by restricting the real log structure of X_0 to X_Δ^0. The most important properties of $X_\Delta^{0,log}(\mathbb{R})$ and $X_{\Delta,1}^{0,log}(\mathbb{R})$, which follow directly from the local description given in Example 3.2.1.1(4), are summarized in the following remark.

Remark 4.2.1.
(1) The morphism $X_\Delta^{0,log}(\mathbb{R}) \to X_\Delta^0(\mathbb{R})$ is a topological cover of degree $2^{|\Delta|}$. In particular $X_\Delta^{0,log}(\mathbb{R})$ is a smooth C^∞-manifold of dimension $\dim(X_\Delta^0)$.
(2) The morphism $X_{\Delta,1}^0(\mathbb{R}) \to X_\Delta^0(\mathbb{R})$ is a topological cover of degree $2^{|\Delta|-1}$. In particular $X_{\Delta,1}^{0,log}(\mathbb{R})$ is a smooth C^∞-manifold of dimension $\dim(X_\Delta^0)$.

By Section 3.3, one has that $\{X_{\Delta,1}^{0,\mathrm{log}}(\mathbb{C})\}$ is a stratification of $X_{0,1}^{log}(\mathbb{C})$. Since $X_{0,1}^{log}(\mathbb{R}) \subseteq X_{0,1}^{log}(\mathbb{C})$ is closed, also $\{X_{\Delta,1}^{0,\mathrm{log}}(\mathbb{R})\}$ is a stratification of $X_{0,1}^{log}(\mathbb{R})$.

4.3. **Equivariant cohomology.** In this section we recall a few generalities on equivariant cohomology as presented in [BBF$^+$60, Chapter IV]. This is an important tool in the proof of Theorem 3.4.2 to compare complex and real Betti numbers. Since [BBF$^+$60, Chapter IV] deals with the case of cohomology and we need the compact cohomology case, we give some details.

In this section we consider $G := \mathbb{Z}/2\mathbb{Z}$ as an abelian group and we let X be a locally compact Hausdorff topological space on which G acts. We write X^G for the set of fixed points for the action of G and X/G for the quotient of X. We assume that $H_c^i(X, \mathbb{Z}/2\mathbb{Z})$ is finite dimensional for all $i \geq 0$ and that there exists some $n_X \in \mathbb{N}$ such that $H_c^i(X, \mathbb{Z}/2\mathbb{Z}) = 0$ for all $i > n_X$. We set

$$b_{i,c}(X) := \dim(H_c^i(X, \mathbb{Z}/2\mathbb{Z})); \quad b_{c,*}(X) := \sum_i b_{i,c}(X).$$

4.3.1. *Group cohomology.* Let M be a finite dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space on which G acts. Write $H^i(G, M)$ for its i^{th}-cohomology group. If M is finite, then $H^i(G, M)$ is finite and there is an inequality $\dim(H^i(G, M)) \leq \dim(M)$. Moreover, if the action of G on M is trivial then $H^i(G, M) = M$ (see e.g. [BBF$^+$60, IV, 2.1, Pag. 50]).

4.3.2. *Classifying space.* We fix an auxiliary integer $N \gg 0$, e.g. $N \geq n_x + 1$. Let $BG := \mathbb{RP}^N$ be the real projective space of dimension N, which is the classifying space for G and for dimension N ([Bor54, Section 18]), in the sense that $\pi_1(BG) \simeq G$ and for every locally constant sheaf $\mathcal{F}$ on BG and every $x \in BG$, one has $H^i(BG, \mathcal{F}) = H^i(G, \mathcal{F}_x)$ for all $i \leq N$. Let $EG := S^N$ be the unit sphere of dimension N, which is the universal cover $EG \to BG$ of BG, endowed with the natural G-action.

[2] We explain how to deduce this from the results of [Arg21] and [Rau22].

- Even if [Arg21, Proposition 7.4] is stated in a very particular case, the proof extends to nice (i.e. exact) proper and logarithmic smooth morphism, since it (explicitly) mimics the proof of Theorem 3.2.2.1;
- [Rau22, Theorem 1.1] is stated for morphisms to a disk. Since an orientation around 0 induces an equivariant isomorphism between a local neighbourhood of 0 with a disk and of a local neighbourhood of 0 in $C^{log}(\mathbb{R})$ with the real oriented blow-up, Rau's result apply to this situation.

4.3.3. *G-equivariant cohomology with compact support.* Define $X_G := (X \times EG)/G$ where G acts on $X \times EG$ diagonally. Then, for every coefficient ring A, the equivariant cohomology of X with compact support and coefficient in A is defined by $H^i_{G,c}(X, A) := H^i_c(X_G, A)$. It is important to keep in mind the following morphisms:

$$X/G \xleftarrow{\;g\;} X_G \xrightarrow{\;h\;} BG,$$

where $g : X_G \to X/G$ (resp. $h : X_G \to BG$) is induced by the first (resp. the second) projection $X \times EG \to X \to X/G$ (resp. $X \times EG \to EG \to BG$).

4.3.4. *Smith-Thom spectral sequence.* The morphism $h : X_G \to BG$ has fibers X, hence the Leray spectral sequence with compact support for h reads

$${}_X E_2^{a,b} := H^a_c(BG, H^b_c(X, A)) \Rightarrow H^{a+b}_{G,c}(X, A),$$

where $H^b_c(X, A)$ is the constant local system associated to the group $H^b_c(X, A)$. Since BG is compact and the classifying space for G and for dimension N, one has $H^a_c(BG, H^b_c(X, A)) = H^a(BG, H^b_c(X, A)) = H^a(G, H^b_c(X, A))$; see Section 4.3.2. Therefore we get a spectral sequence

$$\text{(4.3.4.1)} \qquad {}^{ST}_{X} E_2^{a,b} := H^a(G, H^b_c(X, A)) \Rightarrow H^{a+b}_{G,c}(X, A),$$

which we call the Smith-Thom spectral sequence (with compact support).

4.3.5. *Maximality.* Recall that, by [BBF$^+$60, Application 3.7(b), Pag 54], for $n_X < m < N$, the inclusion $(X^G)_G = X^G \times BG \subseteq X_G$ induces a canonical isomorphism $H^m_{G,c}(X, \mathbb{Z}/2\mathbb{Z}) \simeq H^m_c(X^G \times BG, \mathbb{Z}/2\mathbb{Z}) \simeq H^*_c(X^G, \mathbb{Z}/2\mathbb{Z})$. Since $\dim(H^a(G, H^b_c(X, \mathbb{Z}/2\mathbb{Z})) \leq \dim(H^b_c(X, \mathbb{Z}/2\mathbb{Z}))$, the spectral sequence (4.3.4.1) gives an inequality

$$\text{(4.3.5.1)} \qquad b_{c,*}(X^G) \leq b_{c,*}(X),$$

where $b_{c,*}$ denotes the total Betti number with compact support. We can now recall the definition of maximality in this context.

Definition 4.3.5.2. We say that X is maximal, if $b_{c,*}(X^G) = b_{c,*}(X)$.

The Smith-Thom spectral sequence (4.3.4.1), together with the inequality $|H^a(G, H^b_c(X, \mathbb{Z}/2\mathbb{Z}))| \leq |H^b_c(X, \mathbb{Z}/2\mathbb{Z})|$, implies that X is maximal if and only if ${}^{ST}_{X} E_2^{a,b}$ degenerates at E_2 and the action of G on $H^b_c(X, \mathbb{Z}/2\mathbb{Z})$ is trivial for all b; see e.g. [BBF$^+$60, 4.1, Pag. 55].

4.3.6. *Filtrations and maximality.* Assume now that X is maximal, so that the spectral sequence degenerates at E_2 and the action of G on $H^b_c(X, \mathbb{Z}/2\mathbb{Z})$ is trivial for all b.

Then, for every $n_X < m < N$ the spectral sequence (4.3.4.1) induces a canonical decreasing filtration

$$0 = {}_mF_{m+1} \subseteq {}_mF_m \subseteq \cdots \subseteq {}_mF_{i+1} \subseteq {}_mF_i \subseteq \cdots \subseteq {}_mF_0 = H^*_c(X^G, \mathbb{Z}/2\mathbb{Z})(\simeq H^m_{G,c}(X, \mathbb{Z}/2\mathbb{Z}))$$

such that

$${}_mF_i/{}_mF_{i+1} \simeq {}^{ST}_{X} E_2^{i,m-i} = H^i(G, H^{m-i}_c(X, \mathbb{Z}/2\mathbb{Z})) = H^{m-i}_c(X, \mathbb{Z}/2\mathbb{Z}),$$

where the last equality follows from the fact that the action of G on $H^{m-i}_c(X, \mathbb{Z}/2\mathbb{Z})$ is trivial; see Section 4.3.1.

5. Proof of Theorem 3.4.2(1)

Retain the notation and assumptions of Theorem 3.4.2(1). Up to a choice of an orientation (U, ψ) around $0 \in C(\mathbb{R})$, we can assume that X_t, for some t near 0, is positive; see Section 2.3. In order to simplify the notation, in this section, for every topological space Z, we write

$$H^i(Z) := H^i(Z, \mathbb{Z}/2\mathbb{Z}); \quad H^i_c(Z) := H^i_c(Z, \mathbb{Z}/2\mathbb{Z}); \quad b_{i,c} := \dim(H^i_c(Z)); \quad b_{c,*} := \sum_i b_{i,c}.$$

As explained in Section 1.4 the strategy is the following:

(1) By using the work done in the previous sections, we construct a topological space $X^{log}_{0,1}(\mathbb{C})$ endowed with an involution and with a morphism $X^{log}_{0,1}(\mathbb{C}) \to X_0(\mathbb{C})$ compatible with the involutions, such that $X^{log}_{0,1}(\mathbb{C})$ is homeomorphic to the general smooth fiber of the morphism $X(\mathbb{C}) \to C(\mathbb{C})$ and the set of fixed points $X^{log}_{0,1}(\mathbb{R})$ is homeomorphic to a positive real fiber near 0;

(2) By using the stratification $\{X^0_\Delta\}$ and the geometry of the morphism $X^{log}_{0,1}(\mathbb{C}) \to X_0(\mathbb{C})$, we stratify $X^{log}_{0,1}(\mathbb{C})$ and $X^{log}_{0,1}(\mathbb{R})$ in a compatible way and we transfer the assumptions (a), (b) of Theorem 3.4.2 to these stratifications;

(3) Using the assumption (a) of Theorem 3.4.2 and the spectral sequence for a stratified space, we construct a chain complex ${}^{\mathbb{R}}E_1^{\bullet,0}$ whose cohomology computes the Betti numbers of $X_{0,1}^{log}(\mathbb{R})$;
(4) Using the assumption (b) of Theorem 3.4.2 and the spectral sequence for equivariant cohomology (combined with the assumption (a)), we construct a filtration $F_i^\bullet \subseteq {}^{\mathbb{R}}E_1^{\bullet,0}$ such that $F_i^\bullet/F_{i+1}^\bullet$ is isomorphic to a complex depending only on the complex geometry of $X \to C$;
(5) We conclude the proof by using the spectral sequence for a filtered complex.

5.1. **Construction of $X_{0,1}^{log}(\mathbb{C})$ and $X_{0,1}^{log}(\mathbb{R})$.** We endow $X \to C$ with the real logarithmic structure defined in Section 4.1.2 (see also Sections 3.1.2 and 3.2.2) and recall the commutative diagram

$$\begin{tikzcd} X_{0,1}^{log}(\mathbb{C}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\alpha"] & X_0^{log}(\mathbb{C}) \arrow[r, "\pi"] \arrow[d] & X_0(\mathbb{C}) \\ \{1\} \arrow[r] & 0^{\log}(\mathbb{C}) = S^1, & \end{tikzcd}$$

compatible with the involutions and its real counterpart

$$\begin{tikzcd} X_{0,1}^{log}(\mathbb{R}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\alpha_{\mathbb{R}}"] & X_0^{log}(\mathbb{R}) \arrow[r, "\pi_{\mathbb{R}}"] \arrow[d] & X_0(\mathbb{R}) \\ \{1\} \arrow[r] & 0^{\log}(\mathbb{R}) = \{\pm 1\}. & \end{tikzcd}$$

By Theorems 3.2.2.1 and 4.1.2.1, the general smooth fiber of $X(\mathbb{C}) \to C(\mathbb{C})$ is homeomorphic to $X_{0,1}^{log}(\mathbb{C})$ and the general positive smooth fiber near 0 of $X(\mathbb{R}) \to C(\mathbb{R})$ is homeomorphic to $X_{0,1}^{log}(\mathbb{R})$. Hence, to prove Theorem 3.4.2(1), we can replace $X_t(\mathbb{R})$ with $X_{0,1}^{log}(\mathbb{R})$.

5.2. **Stratifications.** As in Sections 3.3 and 4.2, the collections

$$\{X_{\Delta,1}^{0,log}(\mathbb{C}) := \alpha^{-1}(X_\Delta^0(\mathbb{C}))\} \quad \text{and} \quad \{X_{\Delta,1}^{0,log}(\mathbb{R}) := \alpha_{\mathbb{R}}^{-1}(X_\Delta^0(\mathbb{R}))\}$$

give equivariant stratifications of $X_{0,1}^{log}(\mathbb{C})$ and $X_{0,1}^{log}(\mathbb{R})$ respectively. Recall that, by Remarks 3.3.1 and 4.2.1, one has that

(1) $X_{0,1}^{log}(\mathbb{C})$ is a smooth C^∞-manifold of dimension $2\dim(X_\Delta^0) + |\Delta| - 1$;
(2) $X_{0,1}^{log}(\mathbb{R})$ is a smooth C^∞-manifold of dimension $\dim(X_\Delta^0)$.

The following is the key lemma that allows us to transfer the assumptions (a) and (b) of Theorem 3.4.2 from the stratifications of X_0 to the ones of $X_{0,1}^{\log}(\mathbb{C})$ and $X_{0,1}^{\log}(\mathbb{R})$.

Lemma 5.2.1.
(a) $H^i(X_{\Delta,1}^{0,\log}(\mathbb{R})) = 0$ for $i \geq 1$;
(b) $X_{\Delta,1}^{0,\log}(\mathbb{C})$ is a maximal variety.

Proof. By Remark 4.2.1, one has that $X_{\Delta,1}^{0,\log}(\mathbb{R}) \to X_\Delta^0(\mathbb{R})$ is a topological cover of degree $2^{|\Delta|-1}$. Since

$$H^1(X_\Delta^0(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = 0$$

by assumption, this cover is the trivial one. Hence $X_{\Delta,1}^0(\mathbb{R})$ is the disjoint union of $2^{|\Delta|-1}$ copies of $X_\Delta^0(\mathbb{R})$, so that

$$b_i(X_{\Delta,1}^{0,\log}(\mathbb{C})) = 0 \quad \text{for } i \geq 1 \quad \text{and} \quad b_*(X_{\Delta,1}^{0,\log}(\mathbb{C})) = 2^{|\Delta|-1} b_*(X_\Delta^0(\mathbb{R})), \tag{5.2.2}$$

and, in particular, we get (a).

In order to prove (b), we first observe that, by the Smith-Thom inequality (1.2.1), one has

$$b_*(X_{\Delta,1}^{0,log}(\mathbb{R})) \leq b_*(X_{\Delta,1}^{0,log}(\mathbb{C})),$$

hence it is enough to prove that

$$b_*(X_{\Delta,1}^{0,log}(\mathbb{C})) \leq b_*(X_{\Delta,1}^{0,log}(\mathbb{R})).$$

First of all, one has the following relations

$$b_*(X^{0,log}_{\Delta,1}(\mathbb{R})) = 2^{|\Delta|-1} b_*(X^0_\Delta(\mathbb{R})) = 2^{|\Delta|-1} b_*(X^0_\Delta(\mathbb{C})),$$

where the first equality follows from (5.2.2) and the second one from the assumption (b) of Theorem 3.4.2 on $X^0_\Delta(\mathbb{C})$. Hence, in order to conclude, it is enough show that

$$b_*(X^{0,log}_{\Delta,1}(\mathbb{C})) \leq 2^{|\Delta|-1} b_*(X^0_\Delta(\mathbb{C})).$$

By Remark 3.3.1 the morphism $\alpha : X^{0,\log}_{\Delta,1}(\mathbb{C}) \to X^0_\Delta(\mathbb{C})$ is a locally trivial fibration with fiber $(S^1)^{|\Delta|-1}$. The Leray spectral sequence associated to such morphism α

$$E_2^{a,b} := H^a(X^0_\Delta(\mathbb{C}), R^b\alpha_*\mathbb{Z}/2\mathbb{Z}) \Rightarrow H^{a+b}(X^{0,\log}_{\Delta,1}(\mathbb{C}))$$

shows that

$$b_i(X^{0,\log}_{\Delta,1}(\mathbb{C})) \leq \sum_a \dim(H^a(X^0_\Delta(\mathbb{C}), R^{i-a}\alpha_*\mathbb{Z}/2\mathbb{Z})).$$

By Lemma 5.2.3, that is proven later, the locally constant sheaf $R^i\alpha_*\mathbb{Z}/2\mathbb{Z}$ is constant so that

$$R^i\alpha_*\mathbb{Z}/2\mathbb{Z} \simeq (\mathbb{Z}/2\mathbb{Z})^{\dim(H^i((S^1)^{|\Delta|-1}))}.$$

Hence

$$\begin{aligned}\dim(H^a(X^0_\Delta(\mathbb{C}), R^{i-a}\alpha_*\mathbb{Z}/2\mathbb{Z})) &= \dim(H^a(X^0_\Delta(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}^{\dim(H^{i-a}((S^1)^{|\Delta|-1}))})) = \\ &= b_a(X^0_\Delta(\mathbb{C}))\dim(H^{i-a}((S^1)^{|\Delta|-1})) = b_a(X^0_\Delta(\mathbb{C}))\binom{|\Delta|-1}{i-a}.\end{aligned}$$

This implies that

$$\begin{aligned}\sum_i b_i(X^{0,\log}_{\Delta,1}(\mathbb{C})) &\leq \sum_i\sum_a b_a(X^0_\Delta(\mathbb{C}))\binom{|\Delta|-1}{i-a} = \sum_a \binom{|\Delta|-1}{a}\sum_i b_{i-a}(X^0_\Delta(\mathbb{C})) \\ &\leq \sum_a \binom{|\Delta|-1}{a}\sum_i b_i(X^0_\Delta(\mathbb{C})) = 2^{|\Delta|-1}\sum_a b_a(X^0_\Delta(\mathbb{C})).\end{aligned}$$

□

Lemma 5.2.3. *$R^i\alpha_*\mathbb{Z}/2\mathbb{Z}$ is a constant sheaf.*

Proof. Let the notation be as in the following canonical commutative diagram

$$\begin{tikzcd} X^{0,\log}_{\Delta,1}(\mathbb{C}) \arrow[r, hook, "j"] \arrow[dr, "\alpha"'] & X^{0,\log}_\Delta(\mathbb{C}) \arrow[r, hook, "h"] \arrow[d, "\pi"] & X^{\log}_0(\mathbb{C}) \arrow[d] \\ & X^0_\Delta(\mathbb{C}) \arrow[r, hook] & X_0(\mathbb{C}). \end{tikzcd}$$

First, we show that $R^i\pi_*\mathbb{Z}/2\mathbb{Z}$ is constant. Thanks to [KN99b, Lemma 1.5], one has

$$R^i\pi_*\mathbb{Z}/2\mathbb{Z} \simeq (\bigwedge^i \overline{M}^{gp}_{X^{0,\log}_\Delta}) \otimes \mathbb{Z}/2\mathbb{Z};$$

where the used notation has been introduced in Section 3.1.1. Therefore, it is enough to show that $\overline{M}^{gp}_{X^{0,\log}_\Delta}$ is constant. By [Nak00, Lemma 1.8.1], one has

$$\overline{M}^{gp}_{X^{\log}_0} \simeq \bigoplus_{i\in I}(X_i \to X)_*\mathbb{Z}$$

so that

$$\overline{M}^{gp}_{X^{0,\log}_\Delta} = h^*\overline{M}^{gp}_{X^{\log}_0} = \bigoplus_{i\in I} h^*(X_i \to X)_*\mathbb{Z} = \bigoplus_{i\in I | X^0_\Delta \subseteq X_i}\mathbb{Z}.$$

Now, we deduce that this implies that $R^i\alpha_*\mathbb{Z}/2\mathbb{Z}$ is constant. Since j is a closed immersion and, hence, $R^i\alpha_*\mathbb{Z}/2\mathbb{Z} = R^i\pi_*(j_*\mathbb{Z}/2\mathbb{Z})$, it is enough to show that $R^i\pi_*\mathbb{Z}/2\mathbb{Z} \to R^i\pi_*(j_*\mathbb{Z}/2\mathbb{Z})$ is surjective. This can be checked locally on $X^0_\Delta(\mathbb{C})$. By Example 3.2.1.1(4), we can assume that $X^{0,log}_\Delta(\mathbb{C}) \simeq (S^1)^{|\Delta|} \times X^0_\Delta(\mathbb{C})$ and $X^{0,log}_{\Delta,1}(\mathbb{C})$ is the subset defined by the product of the first $|\Delta|$ components equal to 1. In this case, the result follows from the fact that the inclusion $X^{0,log}_{\Delta,1}(\mathbb{C}) \subseteq X^{0,log}_{\Delta}(\mathbb{C})$ has a (non-canonical) retraction, depending on the choice of one of the first $|\Delta|$ components. □

5.3. **Computations of the real Betti numbers.** Let us consider the natural spectral sequence (see e.g. [Pet16] and [Pet17, (3) pag. 2527])

$$ {}^{\mathbb{R}}E_1^{a,b} := \bigoplus_{\dim(X^0_\Delta)=a} H_c^{a+b}(X^{0,log}_{\Delta,1}(\mathbb{R})) \Rightarrow H_c^{a+b}(X^{log}_{0,1}(\mathbb{R})). \tag{5.3.1}$$

Since $X^{0,log}_{\Delta,1}(\mathbb{R})$ is a smooth C^∞-manifolds of dimension $\dim(X^0_\Delta)$, we can apply Poincaré duality to deduce that

$$\bigoplus_{\dim(X^0_\Delta)=a} H_c^{a+b}(X^{0,log}_{\Delta,1}(\mathbb{R})) \simeq \bigoplus_{\dim(X^0_\Delta)=a} H^{-b}(X^{0,log}_{\Delta,1}(\mathbb{R})).$$

By Lemma 5.2.1(a), we get that ${}^{\mathbb{R}}E_1^{a,b} = 0$ if $b \neq 0$. Hence the first page of the spectral sequence (5.3.1) reduces to the line ${}^{\mathbb{R}}E_1^{\bullet,0}$:

$$\cdots \to \bigoplus_{\dim(X^0_\Delta)=a-1} H_c^{a-1}(X^{0,log}_{\Delta,1}(\mathbb{R})) \xrightarrow{d_{a-1}} \bigoplus_{\dim(X^0_\Delta)=a} H_c^{a}(X^{0,log}_{\Delta,1}(\mathbb{R})) \xrightarrow{d_a} \bigoplus_{\dim(X^0_\Delta)=a+1} H_c^{a+1}(X^{0,log}_{\Delta,1}(\mathbb{R})) \to \dots \tag{5.3.2}$$

so that from (5.3.1) and Theorem 4.1.2.1, one gets

$$H^a_c(X_t(\mathbb{R})) = H^a({}^{\mathbb{R}}E_1^{\bullet,0}), \tag{5.3.3}$$

for t positive and near 0. Hence, in order to conclude the proof of Theorem 3.4.2(1) it is enough to bound the dimension of $H^a({}^{\mathbb{R}}E_1^{\bullet,0})$.

5.4. **Construction of the filtration.** By Lemma 5.2.1(b), one has that $X^{0,log}_{\Delta,1}(\mathbb{C})$ is maximal. The dimension of $X^{0,log}_{\Delta,1}(\mathbb{R})$ is $\dim(X^0_\Delta)$, therefore, since $X^{0,log}_{\Delta,1}(\mathbb{R})$ is a smooth C^∞-variety, we have

$$H_c^{\dim(X^0_\Delta)}(X^{0,log}_{\Delta,1}(\mathbb{R}))) = H_c^*(X^{0,log}_{\Delta,1}(\mathbb{R})).$$

In order to get a filtration of $H_c^{\dim(X^0_\Delta)}(X^{0,log}_{\Delta,1}(\mathbb{R}))$, we use the tools of Section 4.3.6. Once we set $m = 2\dim(X_0)+1$ and choose $N \gg m$, one gets, by using the construction of Section 4.3.6, a decreasing filtration ${}_mF_{\Delta,\bullet}$

$$\cdots \subseteq {}_mF_{\Delta,i+1} \subseteq {}_mF_{\Delta,i} \subseteq \cdots \subseteq {}_mF_{\Delta,0} = H_c^{\dim(X^0_\Delta)}(X^{0,log}_{\Delta,1}(\mathbb{R}))\big(\simeq H^m_{G,c}(X^{0,log}_{\Delta,1}(\mathbb{C})\big)$$

with graded quotients

$$\mathrm{Gr}^i_{{}_mF_{\Delta,\bullet}} := {}_mF_{\Delta,i}/{}_mF_{\Delta,i+1} \simeq H_c^{m-i}(X^{0,log}_{\Delta,1}(\mathbb{C}))$$

Let X^0_Δ and $X^0_{\Delta'}$ be two strata such that $\dim(X^0_\Delta) = a-1 = \dim(X^0_{\Delta'})-1$ and $X^0_\Delta \subseteq X_{\Delta'}$ so that there is a commutative diagram

$$\begin{tikzcd}
H_c^m(X^{0,\log}_{\Delta,1}(\mathbb{C})_G) \arrow[r] & H_c^{m+1}(X^{0,\log}_{\Delta',1}(\mathbb{C})_G) \\
H_c^m(X^{0,\log}_{\Delta,1}(\mathbb{R}) \times BG) \arrow[r] \arrow[u, "\simeq"] & H_c^{m+1}(X^{0,\log}_{\Delta',1}(\mathbb{R}) \times BG) \arrow[u, "\simeq"] \\
\bigoplus_k H_c^{m-k}(X^{0,\log}_{\Delta,1}(\mathbb{R})) \otimes H_c^k(BG) \arrow[r] \arrow[u, "\simeq"] & \bigoplus_k H_c^{m+1-k}(X^{0,\log}_{\Delta',1}(\mathbb{R})) \otimes H_c^k(BG) \arrow[u, "\simeq"] \\
H_c^{a-1}(X^{0,\log}_{\Delta,1}(\mathbb{R})) \otimes H_c^{m-a+1}(BG) \arrow[r] \arrow[u, "\simeq"] & H_c^{a}(X^{0,\log}_{\Delta',1}(\mathbb{R})) \otimes H_c^{m-a+1}(BG) \arrow[u, "\simeq"] \\
H_c^{a-1}(X^{0,\log}_{\Delta,1}(\mathbb{R})) \arrow[r] \arrow[u, "\simeq"] & H_c^{a}(X^{0,\log}_{\Delta',1}(\mathbb{R})), \arrow[u, "\simeq"]
\end{tikzcd}$$

where, for $\# \in \{\Delta, \Delta'\}$,

- the horizontal maps are induced from the stratifications $\{X^{0,\log}_{\Delta,1}(\mathbb{C})_G\}$, $\{X^{0,\log}_{\Delta,1}(\mathbb{R}) \times BG\}$ and $\{X^{0,\log}_{\Delta,1}(\mathbb{R})\}$ of respectively $X^{\log}_{0,1}(\mathbb{C})_G$, $X^{\log}_{0,1}(\mathbb{R}) \times BG$ and $X^{\log}_{0,1}(\mathbb{R})$;

- the upper vertical maps are induced by the inclusions $X^{0,\log}_{1,\#}(\mathbb{R})\times BG \subseteq X^{0,\log}_{1,\#}(\mathbb{C})_G$ and they are isomorphisms by [BBF+60, IV, Pag. 55, Application 3.7(b)], see Section 4.3.5;
- the remaining top-to-bottom vertical maps are induced respectively by the Kunneth formula, the fact that $H^i_c(X^{0,\log}_{1,\#}(\mathbb{R})) = 0$ for $i \neq \dim(X^0_\Delta)$ and $H^{m-a+1}_c(BG) \simeq \mathbb{Z}/2\mathbb{Z}$.

Since the morphisms $H^i_c(X^{0,\log}_{\Delta,1}(\mathbb{C})_G) \to H^{i+1}_c(X^{0,\log}_{\Delta',1}(\mathbb{C})_G)$ and $H^i_c(X^{0,\log}_{\Delta,1}(\mathbb{C})) \to H^{i+1}_c(X^{0,\log}_{\Delta',1}(\mathbb{C}))$ give rise to a morphism of spectral sequences

$$\begin{array}{ccc}
{}^{ST}_{X^{0,\log}_{\Delta,1}(\mathbb{C})}E_2^{a,b} := H^a(G, H^b_c(X^{0,\log}_{\Delta,1}(\mathbb{C}))) & \Longrightarrow & H^{a+b}_c(X^{0,\log}_{\Delta,1}(\mathbb{C})_G) \\
\downarrow & & \downarrow \\
{}^{ST}_{X^{0,\log}_{\Delta',1}(\mathbb{C})}E_2^{a,b+1} := H^a(G, H^{b+1}_c(X^{0,\log}_{\Delta',1}(\mathbb{C}))) & \Longrightarrow & H^{a+b+1}_c(X^{0,\log}_{\Delta',1}(\mathbb{C})_G),
\end{array}$$

the morphism $H^{a-1}_c(X^{0,log}_{\Delta,1}(\mathbb{R})) \to H^a_c(X^{0,log}_{\Delta',1}(\mathbb{R}))$ sends ${}_mF_{\Delta,i}$ to ${}_{m+1}F_{\Delta',i}$.

Therefore, if we set

$${}_mF^p_i := \bigoplus_{\dim(X^0_\Delta)=p} {}_{m+p}F_{\Delta,i}$$

we get a decreasing filtration $F^\bullet_\bullet$ of ${}^{\mathbb{R}}E_1^{\bullet,0}$ (Section 5.3)

$$\begin{array}{ccccc}
{}_mF^{a-1}_{i+1} & \longrightarrow & {}_mF^a_{i+1} & \longrightarrow & {}_mF^{a+1}_{i+1} \\
\downarrow & & \downarrow & & \downarrow \\
{}_mF^{a-1}_{i} & \longrightarrow & {}_mF^a_{i} & \longrightarrow & {}_mF^{a+1}_{i} \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
\cdots \bigoplus_{\dim(X^0_\Delta)=a-1} H^{a-1}_c(X^{0,log}_{\Delta,1}(\mathbb{R})) & \longrightarrow & \bigoplus_{\dim(X^0_\Delta)=a} H^{a}_c(X^{0,log}_{\Delta,1}(\mathbb{R})) & \longrightarrow & \bigoplus_{\dim(X^0_\Delta)=a+1} H^{a+1}_c(X^{0,log}_{\Delta,1}(\mathbb{R}))\cdots
\end{array}$$

such that the graded quotient

$$\mathrm{Gr}^{i,\bullet}_{F_\bullet} := {}_mF^\bullet_i / {}_mF^\bullet_{i+1}$$

is the complex ${}^{\mathbb{C}}E_1^{\bullet,m-i}$ of Section 3.4

$$\begin{array}{ccccc}
{}_mF^{a-1}_i/{}_mF^{a-1}_{i+1} & \longrightarrow & {}_mF^{a}_i/{}_mF^{a}_{i+1} & \longrightarrow & {}_mF^{a+1}_i/{}_mF^{a+1}_{i+1} \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\
\bigoplus_{\dim(X^0_\Delta)=a-1} H^{m+a-i-1}_c(X^{0,log}_{\Delta,1}(\mathbb{C})) & \longrightarrow & \bigoplus_{\dim(X^0_\Delta)=a} H^{m+a-i}_c(X^{0,log}_{\Delta,1}(\mathbb{C})) & \longrightarrow & \bigoplus_{\dim(X^0_\Delta)=a+1} H^{m+a-i+1}_c(X^{0,log}_{\Delta,1}(\mathbb{C})).
\end{array}$$

5.5. **End of the proof.** Recall the spectral sequence for a filtered complex

$${}^FE_1^{r,s} := H^{r+s}(\mathrm{Gr}^{r,\bullet}_{F_\bullet}) \simeq H^{r+s}({}^{\mathbb{C}}E_1^{\bullet,m-r}) \Rightarrow H^{r+s}({}^{\mathbb{R}}E_1^{\bullet,0}).$$

Since

$$H^{r+s}({}^{\mathbb{R}}E_1^{\bullet,0}) \simeq H^{r+s}(X^{\log}_{0,1}(\mathbb{R})) \simeq H^{r+s}(X_t(\mathbb{R})),$$

where the first isomorphism follows from (5.3.3) and second from Theorem 4.1.2.1 (and the compactness of $X_t(\mathbb{R})$), we get that

$$\dim(H^p(X_t(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})) = \sum_q \dim({}^FE_\infty^{q,p-q}) \leq \sum_q \dim({}^FE_1^{q,p-q}) = \sum_q \dim(H^p({}^{\mathbb{C}}E_1^{\bullet,m-q})) = \sum_q \dim(H^p({}^{\mathbb{C}}E_1^{\bullet,q})).$$

6. Proof of Theorem 3.4.2(2)

6.1. **Notation.** Retain the notation and the assumption of Theorem 3.4.2(2). We set

$$X_{0,1}^{\log}(\mathbb{C}) := X_{0,1}^{\log} \quad , \quad X_{\Delta,1}^{0,\log}(\mathbb{C}) := \widetilde{X}_{\Delta}^{0}, \quad X_{\Delta}^{0,\log}(\mathbb{C}) := X_{\Delta}^{0,\log},$$

and for every topological space Z we write $H^i(Z) := H^i(Z, \mathbb{Q})$. Recall the morphisms

$$\pi : X_{0,1}^{\log} \to X_0 \quad \text{and} \quad \pi_\Delta : \widetilde{X}_{\Delta}^{0} \to X_{\Delta}^{0} \tag{6.1.1}$$

and the stratifications

$$\coprod X_{\Delta}^{0} = X_0 \quad \text{and} \quad \coprod \widetilde{X}_{\Delta}^{0} = X_{0,1}^{log}. \tag{6.1.2}$$

For a $\mathbb{Q}$-mixed Hodge structure V, we write Gr_i^W for its weight i-graded piece. Fix $q \in \mathbb{N}$.

6.2. **Strategy.** We detail the strategy to prove Theorem 3.4.2(2).

(1) In order to prove point (ii), what one needs to do is to transfer the assumption on the torsion freeness on the cohomology of X_{Δ}^{0} to the one of $\widetilde{X}_{\Delta}^{0}$. This reduces to study the Leray spectral sequence with integer coefficient for the morphism $\pi_\Delta : \widetilde{X}_{\Delta}^{0} \to X_{\Delta}^{0}$ and to show that it degenerates in E_2; see Lemma 6.4.1.1.
(2) We then deal with point (i). This involves a careful analysis of different spectral sequences coming from geometry. Following [IKMZ19], we first show that the (p, q)-hodge number of the general fiber can be computed as the weight $2q$-part of the limiting Hodge structure on $H^{p+q}(X_{0,1}^{\log})$; see Lemma 6.4.2.3;
(3) The Leray spectral sequence ${}^{L}E_2$ for $X_{0,1}^{\log} \xrightarrow{\pi} X_0$ is a spectral sequence of mixed Hodge structure which degenerates in ${}^{L}E_3$, hence $\mathrm{Gr}_{2q}^{W} H^{p+q}$ can be computed via its second page; see Section 6.3.3;
(4) We then filter the complex ${}^{\mathbb{C}}E_1^{\bullet,q}$ by using the filtration induced on every term by the Leray spectral sequence for $\widetilde{X}_{\Delta}^{0} \xrightarrow{\pi_\Delta} X_{\Delta}^{0}$ and we consider the spectral sequence ${}^{F}E_1$ for the filtered complex; see Section 6.5.1;
(5) Exploiting the assumptions on weights, we show that the first page of ${}^{F}E_1$ is isomorphic, up to change the indexing, to the second page of ${}^{L}E_2$. Hence the cohomology of the lines of ${}^{F}E_1$ computes the weights of the limiting Hodge structure; see Section 6.5.2;
(6) This arguments bounds the cohomology of ${}^{\mathbb{C}}E_1^{\bullet,q}$ with the Hodge numbers of the generic fiber and to get an actual equality, one has to do a few computations recalling how ${}^{\mathbb{C}}E_1^{\bullet,q}$ was constructed; see Section 6.4.

Remark 6.2.1. Another natural strategy to prove Theorem 3.4.2(2)(ii) is to try to reduce to finite fields, by the specialization arguments from [Nak00], and to use the theory of Frobenius weights there. While this might be possible, one issue is that the existence of the spectral sequence with étale cohomology with compact support for logarithmic schemes has not been proved. While this spectral sequence seems likely to exist and it seems to be useful to study it, doing it would take us far away from the techniques used in the rest of this paper. Hence we preferred to remain in the complex analytic setting, at the cost of doing a somehow more convoluted argument.

Before starting with the actual proof of Theorem 3.4.2(ii) in Sections 6.4 and 6.5, we collect some preliminaries on the nearby cycles functors in Section 6.3.

6.3. **Nearby cycles and spectral sequences.**

6.3.1. *Computation of nearby cycles over open strata.* For the lack of a reference, we collect a few results on the nearby cycle functors $R^i\pi_{\Delta,*}A$ for an abelian group A. This is essentially a more refined version of Lemma 5.2.3. The analogue description in the étale case has been done in [Nak00, Appendix A.1].

First, observe that $\pi_{\Delta,*}A \simeq A$, so that we want to compute $R^i\pi_{\Delta,*}A$ for $i \geq 1$. Recall that $\pi_\Delta : \widetilde{X}_{\Delta}^{0} \to X_{\Delta}^{0}$ factorises as

$$\widetilde{X}_{\Delta}^{0} \xrightarrow{\alpha_\Delta} X_{\Delta}^{0,log} \xrightarrow{\beta_\Delta} X_{\Delta}^{0}.$$

Hence, since α_Δ is a closed immersion, one obtains that

$$R^1\pi_{\Delta*}A \simeq R^1\beta_{\Delta,*}\alpha_{\Delta,*}A.$$

Moreover, since $\pi_\Delta : \widetilde{X}_{\Delta}^{0} \to X_{\Delta}^{0}$ is a locally trivial $(S^1)^{|\Delta|-1}$-bundle, the natural cup product map

$$\wedge^i R^1\pi_{\Delta,*}A \xrightarrow{\simeq} R^i\pi_{\Delta,*}A$$

is an isomorphism.

Similar reasonings show that

$$\wedge^i R^1\beta_{\Delta,*}A \xrightarrow{\simeq} R^i\beta_{\Delta,*}A \tag{6.3.1.1}$$

is an isomorphism and, as in Lemma 5.2.3, by [KN99b, Lemma 1.5] and [Nak00], there is a canonical isomorphism

$$R^1\beta_{\Delta,*}A \simeq \bigoplus_{X^0_\Delta \subseteq X_i} A(-1). \tag{6.3.1.2}$$

Working locally as in Lemma 5.2.3, one sees that the natural morphism

$$R^1\beta_{\Delta,*}A \to R^1\pi_{\Delta,*}A \simeq R^1\beta_{\Delta,*}\alpha_{\Delta,*}A$$

is surjective and fits into an exact sequence

$$0 \to A(-1) \xrightarrow{diag} R^1\beta_{\Delta,*}A \simeq \bigoplus_{X^0_\Delta \subseteq X_i} A(-1) \to R^1\pi_{\Delta,*}A \to 0$$

where $diag$ is the diagonal map. Observe that the choice of an i such that $X^0_\Delta \subseteq X_i$, induces a (non-canonical) splitting of the exact sequences. Putting all together, we get a natural exact sequence

$$0 \to A(-1) \otimes R^{i-1}\pi_{\Delta,*}A \to R^i\beta_{\Delta,*}A \to R^i\pi_{\Delta,*}A \to 0$$

which splits (non-canonically). In particular there is a canonical short exact sequence

$$0 \to H^j_c(X^0_\Delta, A(-1) \otimes R^{i-1}\pi_{\Delta,*}A) \to H^j_c(X^0_\Delta, R^i\beta_{\Delta,*}A) \to H^j_c(X^0_\Delta, R^i\pi_{\Delta,*}A) \to 0, \tag{6.3.1.3}$$

which splits (non-canonically).

6.3.2. *Weights and Hodge structures on open strata.* Now by (6.3.1.2) and (6.3.1.1)

$$H^j_c(X^0_\Delta, R^i\beta_{\Delta,*}A) \simeq H^i_c(X^0_\Delta, \wedge^i R^1\beta_{\Delta,*}A) \simeq H^i_c(X^0_\Delta, \wedge^i(\bigoplus_{X^0_\Delta \subseteq X_i} A(-1))) \simeq H^j_c(X^0_\Delta, A(-i))^{\binom{|\Delta|-1}{i}}.$$

Combining this with (6.3.1.3), Poincaré Duality and assumption (c), we get the following.

Lemma 6.3.2.1. *If X^0_Δ has dimension d, then $H^j_c(X^0_\Delta, R^i\pi_*\mathbb{Q})$ carries a natural Hodge structure pure of weight $2j-2d+2i$*

Now, we apply Lemma 6.3.2.1 to the Leray spectral sequences (with compact support)

$${}^L_\Delta E^{a,b}_2 := H^a_c(X^0_\Delta, R^b\pi_{\Delta,*}\mathbb{Q}) \Rightarrow H^{a+b}_c(\widetilde{X}^0_\Delta). \tag{6.3.2.2}$$

associated to the morphisms $\pi_\Delta : \widetilde{X}^0_\Delta \to X^0_\Delta$.

Lemma 6.3.2.3. *${}^L_\Delta E^{a,b}_2$ degenerates in E_2.*

Proof. By Lemma 6.3.2.1, the term ${}^L_\Delta E^{a,b}_2$ is pure of weight $2a - 2\dim(X^0_\Delta) + 2b$. Since there are no morphism between mixed Hodge structures of different weights and the morphisms are compatible with the Hodge structures, the maps ${}^L_\Delta E^{a,b}_r \to {}^L_\Delta E^{a+r,b-r+1}_r$ are trivial for all $a, b, r \in \mathbb{Z}$. □

6.3.3. *Vanishing cycles spectral sequence.* Consider now the Leray spectral sequence for the morphism $\pi : X^{log}_{0,1} \to X_0$

$${}^L E^{a,b}_2 := H^a(X_0, R^b\pi_*\mathbb{Q}) \Rightarrow H^{a+b}(X^{\log}_{0,1}). \tag{6.3.3.1}$$

By [Ill02, Corollary 8.4], the sheaf $R^b\pi_*\mathbb{Q}$ identifies with the classical sheaf of nearby cycles $R^b\Psi\mathbb{Q}$. Hence, by [DS04, Section 1.4], it is a spectral sequence of mixed Hodge structures, called also the vanishing cycles spectral sequence. By [Sch73] (see also [SZ90] and the discussion in [Ill02, Section 8.8, before (8.8.6)]), it degenerates in E_3.

Since taking graded pieces for the weight filtration is an exact operation, we can consider the weight $2q$-graded piece of the spectral sequence (6.3.3.1)

$$\mathrm{Gr}^W_{2q}\, {}^L E^{a,b}_2 := \mathrm{Gr}^W_{2q} H^a(X_0, R^b\pi_*\mathbb{Q}) \Rightarrow \mathrm{Gr}^W_{2q} H^{a+b}(X^{\log}_{0,1}) \tag{6.3.3.2}$$

which still degenerates in E_3, so that

$$\dim(\mathrm{Gr}^W_{2q} H^{p+q}(X^{\log}_{0,1})) = \sum_i \dim(\mathrm{Gr}^W_{2q}\, {}^L E^{i,p+q-i}_3). \tag{6.3.3.3}$$

6.4. **Preliminary reductions.**

6.4.1. *Using torsion freeness.* First, we prove point (ii) in Theorem 3.4.2(2). This is a consequence of the following corollary of Lemma 6.3.2.3, which transfers part of the (c)-assumptions from X^0_Δ to $\widetilde{X}^0_\Delta$.

Corollary 6.4.1.1. *$H^n_c(\widetilde{X}^0_\Delta, \mathbb{Z})$ is torsion free.*

Proof. Let us consider the rational and the integer Leray spectral sequences ${}^L_{\mathbb{Q}}E_2$ and ${}^L_{\mathbb{Z}}E_2$ associated to the morphism $\pi_\Delta : \widetilde{X}^0_\Delta \to X^0_\Delta$. Thanks to Lemma 6.3.2.3, we know that ${}^L_{\mathbb{Q}}E_2$ degenerates in E_2. On the other hand, by hypothesis of Theorem 3.4.2 and the split exact sequence (6.3.1.3), the abelian group ${}^L_{\mathbb{Z}}E_2^{i,n-i}$ has no torsion for all i. Therefore the differential ${}^L_{\mathbb{Z}}E_r^{a,b} \to {}^L_{\mathbb{Z}} E_r^{a+r,b-r+1}$ has to be trivial for all $a, b, r \in \mathbb{N}$, so that ${}^L_{\mathbb{Z}}E_2$ degenerates in E_2. Hence, there exists a decreasing filtration $F^s \subseteq F^{s-1} \subseteq \cdots \subseteq H^n_c(\widetilde{X}^0_\Delta, \mathbb{Z})$ such that $F_i/F_{i+1} \simeq {}^L_{\mathbb{Z}}E_2^{i,n-i}$ has no torsion. This implies that $H^n_c(\widetilde{X}^0_\Delta, \mathbb{Z})$ has no torsion and the proof of the lemma is concluded. □

Thanks to Corollary 6.4.1.1 ${}^{\mathbb{C}}_{\mathbb{Z}}E_1^{\bullet,q} \otimes \mathbb{Z}/2\mathbb{Z} \simeq {}^{\mathbb{C}}_{\mathbb{Z}/2\mathbb{Z}}E_1^{\bullet,q}$ and point (ii) of Theorem 3.4.2(2) is proved.

6.4.2. *Reduction to a weight computation.* Now, we start the proof of (i), by reducing it to a proof of the inequality (6.4.2.2) in the following Lemma.

Lemma 6.4.2.1. *Assume that for all $p, q \in \mathbb{N}$ the inequality*

$$\textit{dim}(H^p({}^{\mathbb{C}}E_1^{\bullet,q})) \leq \textit{dim}(\mathrm{Gr}^W_{2q} H^{p+q}(X^{log}_{0,1})). \tag{6.4.2.2}$$

holds. Then one has $\textit{dim}(H^p({}^{\mathbb{C}}E_1^{\bullet,q})) = h^{p,q}(X_t)$.

In order to prove Lemma 6.4.2.1, we start relating the Hodge numbers of the general fiber to the weight filtration in the cohomology of $X^{\log}_{0,1}$.

Lemma 6.4.2.3. *One has the equality $h^{p,q}(X_t) = \dim(\mathrm{Gr}^W_{2q} H^{p+q}(X^{\log}_{0,1}))$ and the mixed Hodge structure on $H^{p+q}(X^{\log}_{0,1})$ has only even weights.*

Proof. This is essentially proved in [IKMZ19, Middle of Page 31]. We briefly recall the argument. For every subset $J \subseteq I$ consider the spectral sequence

$${}^{\mathbb{C}}E_1^{a,b} := \bigoplus_{\substack{X^0_\Delta \subseteq X_J,\\ \dim(X^0_\Delta)=a}} H^{a+b}_c(X^0_\Delta) \Rightarrow H^{a+b}(X_J).$$

By assumption and Poincaré duality, ${}^{\mathbb{C}}E_1^{a,b}$ is pure of type (b,b), hence $H^n(X_J) = 0$ for n odd and $H^{2n}(X_J) \simeq \mathbb{Q}(-n)^{n_J}$ for some $n_J \in \mathbb{Z}$. Hence, the weight spectral sequence

$$E_2^{i,j} := \bigoplus_{a \geq \max\{0,i\}} \bigoplus_{|J|=2a-i} H^{2i+j-2a}(X_J)(i-a) \Rightarrow H^{i+j}(X^{\log}_{0,1})$$

shows that $W^{2i} = F_{p+q-i}$, where W^j is the increasing weight filtration on $H^{p+q}(X^{\log}_{0,1})$ and F_j the decreasing Hodge filtration. Hence

$$h^{p,q}(X_t) = h^{q,p}(X_t) = \dim(F_p/F_{p+1}) = \dim(W^{2q}/W^{2q-2}) = \dim(\mathrm{Gr}^W_{2q} H^{p+q}(X^{\log}_{0,1})).$$

□

Proof of Lemma 6.4.2.1. Lemma 6.4.2.3 implies that

$$\dim(H^{p+q}(X^{log}_{0,1})) = \sum_i \mathrm{Gr}^W_{2i} H^{p+q}(X^{log}_{0,1}).$$

Therefore, by the spectral sequence (3.4.1), one gets the inequality

$$\sum_i \dim(H^i({}^{\mathbb{C}}E_1^{\bullet,p+q-i})) \geq \sum_i \dim(\mathrm{Gr}^W_{2i} H^{p+q}(X^{log}_{0,1})).$$

The assumption (6.4.2.2) implies that

$$\dim(H^p({}^{\mathbb{C}}E_1^{\bullet,q})) = \mathrm{Gr}^W_{2q} H^{p+q}(X^{log}_{0,1}).$$

Hence, again by Lemma 6.4.2.3, one has $\dim(H^p({}^{\mathbb{C}}E_1^{\bullet,q})) = h^{p,q}(X_t)$. □

6.5. **End of the proof.** By Lemma 6.4.2.1, in order to end the proof of Theorem 3.4.2, it remains to prove the inequality (6.4.2.2).

6.5.1. *Filtering the complex.* Recall the spectral sequences

$$ {}^r_qE_1^{a,b} := \bigoplus_{\dim(X^0_\Delta)=a} H_c^{a+b}(X^0_\Delta, R^{q-r}\pi_{\Delta,*}\mathbb{Q}) \Rightarrow H^{a+b}(X_0, R^{q-r}\pi_*\mathbb{Q}) $$

associated to the stratifications (6.1.2). By Lemma 6.3.2.3, there exists a decreasing filtration ${}_nF_{i,\Delta}$ of $H^n_c(\widetilde{X}^0_\Delta)$ such that ${}_nF_{i,\Delta}/{}_nF_{i+1,\Delta} \simeq H^i_c(X^0_\Delta, R^{n-i}\pi_{\Delta,*}\mathbb{Q})$.

If $\dim(X^0_\Delta)$ and $\dim(X^0_{\Delta'})$ are respectively $m-1$ and m, and $X^0_\Delta \subseteq X_{\Delta'}$, there is a morphism of spectral sequences

$$\begin{array}{ccc} {}^L_\Delta E_2^{a,b} := H^a_c(X^0_\Delta, R^b\pi_{\Delta,*}\mathbb{Q}) & \Longrightarrow & H_c^{a+b}(\widetilde{X}^0_\Delta) \\ \downarrow & & \downarrow \\ {}^L_{\Delta'}E_2^{a+1,b} := H^{a+1}_c(X^0_{\Delta'}, R^b\pi_{\Delta',*}\mathbb{Q}) & \Longrightarrow & H_c^{a+b+1}(\widetilde{X}^0_{\Delta'}), \end{array}$$

so that the morphism $H^{n-1}_c(\widetilde{X}^0_\Delta) \to H^n_c(\widetilde{X}^0_{\Delta'})$ sends ${}_nF_{\Delta,i}$ to ${}_{n+1}F_{\Delta',i+1}$. Hence, setting

$$ {}^qF^p_i := \bigoplus_{\dim(\widetilde{X}^0_\Delta)=p} {}_{p+q}F_{\Delta',i+p} \subseteq \bigoplus_{\dim(\widetilde{X}^0_\Delta)=p} H_c^{p+q}(\widetilde{X}^0_\Delta) = {}^{\mathbb{C}}E_1^{p,q}, $$

we get a decreasing filtration

$$ {}^qF^\bullet_\bullet := \quad {}^qF^\bullet_{i+1} \subseteq^q F^\bullet_i \subseteq \cdots \subseteq {}^qF^\bullet_0 = {}^{\mathbb{C}}E_1^{\bullet,q} $$

such that

$$ \mathrm{Gr}^{i,\bullet}_{{}^qF_\bullet} := {}^qF^\bullet_i/{}^qF^\bullet_{i+1} \simeq {}^i_qE_1^{\bullet,i}. $$

Hence the spectral sequence for a filtered complex reads

$$ {}^F_qE_1^{a,b} = H^{a+b}(\mathrm{Gr}^{a,\bullet}_{{}^qF_\bullet}) = H^{a+b}({}^a_qE_1^{\bullet,a}) = {}^a_qE_2^{a+b,a} \Rightarrow H^{a+b}({}^{\mathbb{C}}E_1^{\bullet,q}). \tag{6.5.1.1}$$

6.5.2. *Computation of the spectral sequence of the filtered complex.* By (6.5.1.1)

$$ \dim(H^p({}^{\mathbb{C}}E^{\bullet,q})) \le \sum_i \dim({}^F_qE_2^{i,p-i}). \tag{6.5.2.1}$$

Now, we fully employ the assumption on the weights by using Lemma 6.3.2.1.

Lemma 6.5.2.2. *For every $r \in \mathbb{N}$ the spectral sequence*

$$ {}^r_qE_1^{a,b} := \bigoplus_{dim(X^0_\Delta)=a} H_c^{a+b}(X^0_\Delta, R^{q-r}\pi_{\Delta,*}\mathbb{Q}) \Rightarrow H^{a+b}(X_0, R^{q-r}\pi_*\mathbb{Q}), $$

degenerates in E_2 and it induces a natural isomorphism

$$ {}^r_qE_2^{a,b} \simeq \mathrm{Gr}^W_{2b+2q-2r} H^{a+b}(X_0, R^{q-r}\pi_*\mathbb{Q}). $$

Proof. Since $H_c^{a+b}(X^0_\Delta, R^{q-r}\pi_{\Delta,*}\mathbb{Q})$ is pure of weight $2b+2q-2r$ by Lemma 6.3.2.1 and there are no morphisms between Hodge structures of different weights, the spectral sequence degenerates in E_2 and it induces the displayed isomorphism. □

By (6.5.1.1) one has

$$ {}^F_qE_1^{a,b} = {}^a_qE_2^{a+b,a}, \tag{6.5.2.3}$$

which by Corollary 6.5.2.2, is canonically isomorphic to

$$ {}^a_qE_2^{a+b,a} \simeq \mathrm{Gr}^W_{2q} H^{2a+b}(X_0, R^{q-a}\pi_*\mathbb{Q}), $$

which by definition (see (6.3.3.2)) is $\mathrm{Gr}^W_{2q}\ {}^LE_2^{2a+b,q-a}$. Hence the bijection $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ sending (x,y) to $(2x+y, q-x)$ (with inverse $(x,y) \mapsto (q-y, 2y+x-2q)$), induces a natural isomorphism

$$ {}^F_qE_1^{a,b} \simeq \mathrm{Gr}^W_{2q}\ {}^LE_2^{2a+b,q-a}. \tag{6.5.2.4}$$

So one gets

$$ \dim(H^p({}^{\mathbb{C}}E_1^{\bullet,q})) \le \sum_i \dim({}^F_qE_2^{i,p-i}) = \sum_i \dim(\mathrm{Gr}^W_{2q}\ {}^LE_3^{p+i,q-i}) = \sum_i \dim(\mathrm{Gr}^W_{2q}\ {}^LE_\infty^{p+i,q-i}) = \dim(\mathrm{Gr}^W_{2q} H^{p+q}(X^{log}_{0,1})), $$

where, to summarize, the first inequality comes from (6.5.1.1), the first equality from (6.5.2.4), the second one from the degeneration of $\mathrm{Gr}^W_{2q}\ {}^LE$ in E_3 (Section 6.3.3) and the last one from (6.3.3.1). Hence we proved the inequality (6.4.2.2) and the proof of Theorem 3.4.2(2) is concluded.

REFERENCES

[AB21] H. Argüz and P. Bousseau. Real log curves in toric varieties, tropical curves, and log Welschinger invariants. *Ann. Inst. Fourier (Grenoble)*, 72(4):1547–1620, 2021.

[Arg21] H. Argüz. Real loci in (log) Calabi-Yau manifolds via Kato-Nakayama spaces of toric degenerations. *Eur. J. Math.*, 7(3):869–930, 2021.

[BBF$^+$60] A. Borel, G. Bredon, E. E. Floyd, D. Montgomery, and R. Palais. *Seminar on Transformation Groups. (AM-46)*. Princeton University Press, 1960.

[BdMR22] E. Brugallé, L. López de Medrano, and J. Rau. Combinatorial patchworking: back from tropical geometry. *arXiv:2209.14043*, 2022.

[Bor54] A. Borel. Sur l'Homologie et la Cohomologie des Groupes de Lie Compacts Connexes. *American Journal of Mathematics*, 76(2):273–342, 1954.

[Bru22] E. Brugallé. Euler characteristic and signature of real semi-stable degenerations. *Journal of the Institute of Mathematics of Jussieu*, 21(1):1–8, 2022.

[DS04] A. Dimca and M. Saito. Some consequences of perversity of vanishing cycles. *Annales de l'Institut Fourier*, 54(6):1769–1792, 2004.

[FFR21] S. Felten, M. Filip, and H. Ruddat. Smoothing toroidal crossing spaces. *Forum of Mathematics, Pi*, 9:E7, 2021.

[GB10] M. Gross and B.Siebert. Mirror symmetry via logarithmic degeneration data, II. *J. Algebraic Geom.*, 19:679–780, 2010.

[IKMZ19] I. Itenberg, L. Katzarkov, G. Mikhalkin, and I. Zharkov. Tropical homology. *Math. Ann.*, 374:963–1006, 2019.

[Ill02] L. Illusie. An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology. In Berthelot Pierre, Fontaine Jean-Marc, Illusie Luc, Kato Kazuya, and Rapoport Michael, editors, *Cohomologies p-adiques et applications arithmétiques (II)*, number 279 in Astérisque. Société mathématique de France, 2002.

[Ite17] I. Itenberg. Tropical homology and Betti numbers of real algebraic varieties. *https://web.ma.utexas.edu/users/sampayne/pdf/Itenberg-Simons2017.pdf*, 2017.

[Kat89] K. Kato. Logarithmic structures of Fontaine-Illusie. *Algebraic Analysis, Geometry and Number Theory*, 1989.

[KN99a] K. Kato and C. Nakayama. Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over $\mathbb{C}$. *Kodai Mathematical Journal*, 22(2):161 – 186, 1999.

[KN99b] K. Kato and C. Nakayama. Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over $\mathbf{C}$. *Kodai Mathematical Journal*, 22(2):161 – 186, 1999.

[Nak00] C. Nakayama. Degeneration of ℓ-Adic Weight Spectral Sequences. *American Journal of Mathematics*, 122(4):721–33, 2000.

[NO10] C. Nakayama and A. Ogus. Relative rounding in toric and logarithmic geometry. *Geometry and Topology*, 14(4):2189 – 2241, 2010.

[Par98] A. Parusiński. Blow-analytic retraction onto the central fibre. *Real analytic and algebraic singularities*, 381:43–61, 1998.

[Pet16] D. Petersen. A spectral sequence for computing cohomology of a space from that of its strata. *https://mathoverflow.net/questions/233067/a-spectral-sequence-for-computing-cohomology-of-a-space-from-that-of-its-strata/233103233103*, 2016.

[Pet17] D. Petersen. A spectral sequence for stratified spaces and configuration spaces of points. *Geometry and Topology*, 21:2527–2555, 2017.

[Rau22] J. Rau. Real semi-stable degenerations, real-oriented blow-ups and straightening corners. *arXiv:2203.17097*, 2022.

[RRS23] J. Rau, A. Renaudineau, and K. Shaw. $Realphasestructuresontropicalmanifoldsandpatchworksinhighercodimension.$ *arxiv: 2310.08313*, 2023.

[RS23] A. Renaudineau and K. Shaw. Bounding the Betti numbers of real hypersurfaces near the tropical limit. *Ann. Scient. Éc. Norm. Sup.*, 56(4e série):945–980, 2023.

[RSTZ14] H. Ruddat, N. Sibilla, D. Treumann, and E. Zaslow. Skeleta of affine hypersurfaces. *Geom. Topol.*, 18(3):1343–1395, 2014.

[Sch73] Wilfried Schmid. Variation of Hodge Structure: The Singularities of the Period Mapping. *Inventiones Mathematicae*, 22:211–320, 1973.

[Ste76] J. Steenbrink. Limits of hodge structures. *Inventiones mathematicae*, 31:229–258, 1976.

[SZ90] Morihiko Saito and Steven Zucker. The kernel spectral sequence of vanishing cycles. *Duke Mathematical Journal*, 61(2):329 – 339, 1990.

[Tho15] R. Thom. Sur l'homologie des variétés algébriques réelles. *Differential and Combinatorial Topology: A Symposium in Honor of Marston Morse*, pages 255–265, 2015.

[Vir83] O. Y. Viro. Gluing of algebraic hypersurfaces, smoothing of singularities and con- struction of curves. *Proc. Leningrad Int. Topological Conf.*, (Leningrad, Aug. 1983). Nauka, Leningrad (Russian).:149 – 197, 1983.

[Vir84] O. Y. Viro. Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7. In *Topology (Leningrad, 1982)*, volume 1060 of *Lecture Notes in Math.*, pages 187–200. Springer, Berlin, 1984.

Emiliano Ambrosi, UNIVERSITÉ DE STRASBOURG
eambrosi'at'unistra.fr: Institut de Recherche Mathématique Avancée (IRMA)

Matilde Manzaroli, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, 72076 TÜBINGEN, GERMANY
matilde.manzaroli'at'uni-tuebingen.de: Mathematisches Institut