# Riemann surfaces and algebraic curves, Exercise sheets 1-13

## 1 Exercise sheet 1

(Exercise 1) (ex. 2.4.4 [1]) Let $K$ be a Klein bottle and $T$ be a torus. Show that

$$K = \mathbb{P}^2(\mathbb{R})\#\mathbb{P}^2(\mathbb{R}) \text{ and } T\#\mathbb{P}^2(\mathbb{R}) = \mathbb{P}^2(\mathbb{R})\#\mathbb{P}^2(\mathbb{R})\#\mathbb{P}^2(\mathbb{R}).$$

Remember that $K$ has identification polygon $aba^{-1}b$.

(Exercise 2) (ex. 2.4.6 [1]) Let $S$ be a compact, connected surface, represented by an identification polygon $w$. Show that the boundary of the polygon $w$ is a *good graph*.

(Exercise 3) (ex. 2.4.7 [1]) Let $S_1$ and $S_2$ be compact, connected surfaces. Show that

$$\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2,$$

where $\chi(*)$ denotes the Euler characteristic.

(Exercise 4) (ex. 2.4.8 [1]) Show that $\chi(T^{\#g}) = 2 - 2g$ and $\chi(\mathbb{P}^2(\mathbb{R})^{\#m}) = 2 - m$.

(Exercise 5) (Euler's identity)

(i) Let $P(x_0, x_1, x_2)$ be a homogeneous polynomial of degree $d$ in variables $x_0, x_1, x_2$. Show that

$$\sum_{i=0}^{2} x_i \frac{\partial P(x_0, x_1, x_2)}{\partial x_i} = dP(x_0, x_1, x_2).$$

(ii) Write $P(x_0, x_1, x_2)$ with respect to its second partial derivatives $\partial_{x_i x_j} P := \frac{\partial \frac{\partial P(x_0,x_1,x_2)}{\partial x_i}}{\partial x_j}$.

## 2 Exercise sheet 2

Guided exercises about blow-ups. Blow-ups can be used, for example, to desingularise real plane curves. Let $C$ be an algebraic plane curve in $\mathbb{P}^2_{\mathbb{C}}$. The aim of this Exercise sheet is to find a compact Riemann surface $X$ and a map $f : X \to \mathbb{P}^2_{\mathbb{C}}$ inducing an isomorphism between $X \setminus f^{-1}(C_{sing})$ and $C \setminus (C_{sing})$, where $C_{sing}$ denotes the singular point set of $C$.

(Exercise 1) Let's start with a local construction. Let $U$ be a neighbourhood of a point $p = (0,0) \in \mathbb{C}^2$ and denote with $z, w$ the coordinates of the affine plane $\mathbb{C}^2$. We want to define the *blow-up of $U$ at $p$*. Take

$$\tilde{U} = \{((z,w),[\xi,\eta]) \in U \times \mathbb{P}^1_{\mathbb{C}} | z\eta = w\xi\}.$$

(i) Let us show that $\tilde{U}$ is a dimension 2 complex variety.
Clearly $\tilde{U}$ is a closed set of $U \times \mathbb{P}^1_{\mathbb{C}}$. Put on $\tilde{U}$ the induced topology. One can cover $\tilde{U}$ with two open sets

$$\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$$

where

$$\tilde{U}_0 = \{((z,w),[\xi,\eta]) \in \tilde{U} | \eta \neq 0\} \cong \mathbb{C}^2,$$
$$\tilde{U}_0 = \{((z,w),[\xi,\eta]) \in \tilde{U} | \xi \neq 0\} \cong \mathbb{C}^2.$$

Let $\pi : \tilde{U} \to U$ be the projection sending $((z,w),[\xi,\eta])$ to $(z,w)$.

- Write the coordinates of $\tilde{U}_0$ and $\tilde{U}_1$ and express $\pi_{|_{\tilde{U}_i}}$ for $i = 0,1$.
- Does it follow from the previous points dealt with in $(i)$ that $(\tilde{U}_i, \pi_{|_{\tilde{U}_i}})_{i=0,1}$ form an atlas for $\tilde{U}$? If not, find an atlas $(\tilde{U}_i, \tilde{\phi}_i)_{i=0,1}$.

(ii) Now let us understand $\tilde{U}$.

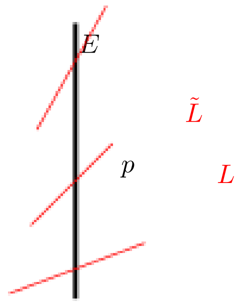
- Show that $\tilde{U}$ contains a copy $E$ of $\mathbb{P}^1_{\mathbb{C}}$.

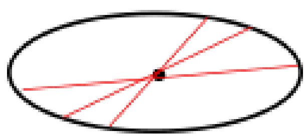
Let $L$ be a line in $U$ of equation $az + bw = 0$ and consider the *strict transform* $\tilde{L}$ of $L$ which is defined as follows:

$$\tilde{L} = \overline{\pi^{-1}(L \setminus \{p\})}.$$

- Show that $\tilde{L} = \{((z,w),[b,-a])\}$.
- Show that $\pi^{-1}(L) = \tilde{L} \cup E$ and describe the set of points of $\tilde{L} \cap E$.

From the previous points dealt with in $(ii)$, it follows that the map $L \mapsto \tilde{L} \cap E$ gives a bijection between points of $E$ and lines through the origin of $\mathbb{C}^2$.



(Exercise 2) Let us consider the case in which $U = \mathbb{C}^2$. Let $C$ be an algebraic curve of equation $zw + p(z, w) = 0$, where $p(z, w)$ is a linear combination of monomials of degree at least 3. Assume that the only singularity of $C$ is at $p = (0, 0)$. Let us consider the strict transform

$$X = \overline{\pi^{-1}(C \setminus \{p\})}.$$

- Write $X \cap \tilde{U}_i$ for $i = 0, 1$ and the intersection points of $X \cap E = \pi^{-1}(p)$.
- Is $X$ is a complex subvariety of $\tilde{U}$?
- Explain why $\pi : X \setminus \pi^{-1}(p) \to C \setminus \{p\}$ is an isomorphism.

$X$ is the desingularization of $C$, i.e. $X$ has no more singularities.

(Exercise 3) Let us consider the *cuspidal* cubic $C$ in $\mathbb{C}^2$ with equation $w^2 + z^3 = 0$. Show that $C$ is desingularised blowing-up $\mathbb{C}^2$ at the origin, i.e. the strict transform of $C$ via $\pi$ has no more singularities. Is the curve of equation $w^2 + z^5 = 0$ in $\mathbb{C}^2$ desingularised after blowing-up the origin? Explain your answer.

## 3 Exercise sheet 3

(Exercise 1) Let $C = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = x^d + y^d + z^d = 0\}$ be a plane curve in $\mathbb{P}^2_{\mathbb{C}}$ (as in *Exercise session 1*). Take the projection map $\pi : C \to \mathbb{P}^1_{\mathbb{C}}$ sending $[x : y : z]$ to $[x : y]$.

- Is $\pi$ an holomorphic map?
- Find all ramification and branch points of $\pi$.
- Compute the genus of $C$ using Riemann-Hurwitz formula.

(Exercise 2) Consider $f : E \to \mathbb{P}^1(\mathbb{C})$, with $g_E = 2$. Assume that there are two points $x_1, x_2 \in E$ where $f$ is fully ramified. How many ramification points can $f$ have, and what ramification is possible?

(Exercise 3) Let $f : X \to Y$ be a holomorphic map of compact Riemann surfaces of degree $d > 0$ and let $y \in Y$ with $f^{-1}(y) = \{x_1, \ldots, x_n\}$. Show that

$$\sum_{i=1}^{n} k_{x_i} = d.$$

Hint: generalise Theorem 1.4.9 on the notes.
**Remark**: Given a point $x$ in $X$, the ramification index $k_x$ is well defined. Namely, if $f$ has two local expressions around $x$ of the form $F(z) = z^k$ and $\tilde{F}(z) = z^{\tilde{k}}$, then $k = \tilde{k}$.

## 4 Exercise sheet 4

(Exercise 1) Let $n$ be a postitive integer. Let $C_0$ be an affine curve

$$\{(x, y) \in \mathbb{C}^2 | y^{n-1} = x^n - 1\}$$

and let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be the projective closure of $C_0$.

- Show that $C$ is non-singular.
- Compute the genus of $C$ applying Riemann-Hurwitz's formula to the function $x$.
- Compute the genus of $C$ applying Riemann-Hurwitz's formula to the function $y$.

(Exercise 2) Let $L$ be a lattice in $\mathbb{C}$. Let $m \in \mathbb{C}_{\neq 0}$. Show that $mL$ is a lattice in $\mathbb{C}$ and that

$$\phi : \mathbb{C}/L \to \mathbb{C}/mL$$

sending $[z]$ to $[mz]$ is a well defined biholomorphic map.

## 5 Exercise sheet 5

Let $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ be a lattice in $\mathbb{C}$ and let $\wp$ be the Weierstrass $\wp$-function for the lattice $\Lambda$. Show that

(Exercise 1) $\wp'$ has degree 3, and it has three distinct simple zeros in $\mathcal{D} = \{s\tau_1 + t\tau_2 : 0 \leq s, t < 1\}$ and those are $\frac{\tau_1}{2}, \frac{\tau_2}{2}, \frac{\tau_3}{2}$, where $\tau_1 + \tau_2 = \tau_3$.

(Exercise 2) the function $z \mapsto \wp(z) - c$ for each choice of $c \in \mathbb{C}$ has a double zero in $z_0 \in \mathcal{D}$ if and only if $c \in \{u_1, u_2, u_3\}$ where $u_i = \wp(\frac{\tau_i}{2})$. Moreover the $u_i$ are three distinct complex numbers.

## 6 Exercise sheet 6

Let $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ be a lattice in $\mathbb{C}$ and let $\wp$ be the Weierstrass $\wp$-function for the lattice $\Lambda$. Let $g_2, g_3 \in \mathbb{C}$ such that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.

(Exercise 1) Show that the subset

$$C = \{[x : y : h : t] \in \mathbb{P}^3_{\mathbb{C}} : y^2 = 4xt - g_2xh - g_3h^2, x^2 = ht\},$$

is a submanifold of $\mathbb{P}^3_{\mathbb{C}}$.
Hint: Implicit Function Theorem.

(Exercise 2) Show that the map

$$\mathbb{C}/\Lambda := T \to \mathbb{P}^3_{\mathbb{C}},$$

sending $z \mapsto [\wp(z) : \wp'(z) : 1 : \wp^2(z)]$ for $z \neq 0$ and $0 \mapsto [0 : 0 : 0 : 1]$ is a holomorphic map.

## 7 Exercise sheet 7

Let $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ be a lattice in $\mathbb{C}$.

(1) Describe the 3-torsion points of $\mathbb{C}/\Lambda$.

(2) How many are they?

(3) Let $\phi$ be the Weierstrass embedding with respect to $\Lambda$. Let $E$ be $\phi(\mathbb{C}/\Lambda)$ and $P \in E$. Show that if $3P = 0$ then the Hessian polynomial of $E$ vanishes at $P$, where the Hessian polynomial of $E$ is the determinant of the Hessian matrix of $E$.

(4) Show that the 3-torsion points of $\mathbb{C}/\Lambda$ correspond to 3-tangents of $E$, where a 3-tangent of $E$ is a line intersecting $E$ in only one point.

## 8 Exercise sheet 8

(Exercise 1) Let $\Lambda$ be a lattice generated by 1 and $\tau$. Let $\xi = a\tau + b$ and $\lambda = p\tau + q$ where $a, b, p, q \in \mathbb{Z}$ (same hypothesis of Theorem 3.3.12/proof of Theorem 3.3.12 in the notes). Show that $\Theta_\xi(z + \lambda) = e_\xi(\lambda, z)\Theta_\xi(z)$ explicitly computing $e_\xi(\lambda, z)$.

(Exercise 2) Let $C := \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = 0\}$, where $F(x, y, z)$ is a non singular polynomial of degree $d$, i.e. $(F, \partial F/\partial x, \partial F/\partial y, \partial F/\partial z) \neq (0, 0, 0, 0)$. Let $\phi(x, y, z)$ be a homogenous non-singular polynomial of degree $d - 3$. Set

$$\omega = \frac{\phi(x, y, z)z^2}{\frac{\partial F}{\partial y}} d(\frac{x}{z}).$$

Study the differential $\omega$ in $C \cap \{z \neq 0\}$ and determine whether $\omega$ is holomorphic in $C \cap \{z \neq 0\}$.

## 9 Exercise sheet 9

(Exercise 1) Let $C := \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = 0\}$, where $F(x, y, z)$ is a non singular polynomial of degree 4, i.e. $(F, \partial F/\partial x, \partial F/\partial y, \partial F/\partial z) \neq (0, 0, 0, 0)$. Let $r$ be a line in $\mathbb{P}^2_{\mathbb{C}}$ given by the equation $L(x, y, z) = 0$. Define a divisor $D$ on $C$ as the intersection of $r$ with $C$.

- Show that $deg(D) = 4$.
- Show that any other divisor $D' \in |D|$ is of the form $D' = r'.C$, for some line $r'$ in $\mathbb{P}^2_{\mathbb{C}}$ of equation $L'(x, y, z) = 0$.

(Exercise 2) Let $C := \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = 0\}$, where $F(x, y, z)$ is a non singular polynomial of degree $d$. Let $\Gamma = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : G(x, y, z) = 0\}$ be a degree $(d-3)$ non singular curve. Define the intersection divisor of $\Gamma$ and $C$ as follows:

$$\Gamma.C = \sum_{p \in C} I(p, \Gamma \cap C)p$$

where $I(p, \Gamma \cap C) = ord_p(\frac{G}{H})$ with $H$ a homogeneous polynomial of degree $(d-3)$ non vanishing at $p$. Remark that $\frac{G}{H}|_C$ is a meromorphic function on $C$.

- Show that if $p \notin C \cap \Gamma$, then $I(p, \Gamma \cap C) = 0$.
- Show that the definition of $I(p, \Gamma \cap C)$ (namely the definition of $\Gamma.C$) does not depend on the choice of $H$.

## 10 Exercise sheet 10

From **Exercise Sessions 3, 6, 8**:

(Exercise 1) Let us consider the curve

$$\Gamma = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} | xyz^3 + x^5 + y^5 = 0\}.$$

Let $\sigma : C \to \Gamma$ be the desingularisation of $\Gamma$. Let $p = [0 : 0 : 1]$.

- Show that $\sigma^{-1}(p)$ consists of two points $\sigma^{-1}(p) = \{p_1, p_2\}$.
- Let $f = x \circ \sigma$ and $h = y \circ \sigma$. Describe zeros, poles, ramification order, ect... of $f, h$ and $f/h$.

(Exercise 2) Let $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ be a lattice in $\mathbb{C}$ and let $\wp$ be the Weierstrass $\wp$-function for the lattice $\Lambda$.
Let $g_2, g_3 \in \mathbb{C}$ such that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. The subset

$$C = \{[x : y : h : t] \in \mathbb{P}^3_{\mathbb{C}} : y^2 = 4xt - g_2xh - g_3h^2, x^2 = ht\},$$

is a submanifold of $\mathbb{P}^3_{\mathbb{C}}$.
The map

$$\mathbb{C}/\Lambda := T \to \mathbb{P}^3_{\mathbb{C}},$$

sending $z \mapsto [\wp(z) : \wp'(z) : 1 : \wp^2(z)]$ for $z \neq 0$ and $0 \mapsto [0 : 0 : 0 : 1]$ is a holomorphic map. Moreover $\phi(T) = C$.

- Show that the map $\phi$ is injective and has maximal rank 1 on $T$.

(Exercise 3) Let $D$ be a divisor on a complex torus $X := \mathbb{C}/\Lambda$ with $deg(D) > 0$. Then $dim(\mathcal{L}(D)) = deg(D)$.

## 11 Exercise sheet 11

The following are exercises $A$ and $E$ from [2]: Algebraic curves and Riemann surfaces, Problems $VI.2$.

(Exercise 1) Given a meromorphic function $f$, and a divisor $D$, show that the multiplication operator

$$\mu_f^D : \mathcal{T}[D](X) \to \mathcal{T}[D - div(f)](X),$$

defined by sending $\sum r_p.p$ to the suitable truncation of $\sum(fr_p).p$ is an isomorphism. Show that the inverse is $\mu_{1/f}^{D-div(f)}$.

(Exercise 2) Let X be the Riemann sphere. Show that $H^1(0) = 0$ by explicitly finding a preimage under $\alpha_0$ for any Laurent tail divisor $Z$ in $\mathcal{T}[0](X)$. (Hint: use partial fractions).

## 12 Exercise sheet 12

Let $D$ denote a divisor on an algebraic curve $X$ of genus $g$.

(Exercise 1) [2, Problems VI.3 B.] Show that if $D$ is a positive divisor of degree at least $g+1$, then there is a non-constant function in $\mathcal{L}(D)$.

(Exercise 2) Show that if $dim\mathcal{L}(D) \neq 0$, then $deg(D) \geq 0$. Moreover show that if $dim\mathcal{L}(D) \neq 0$ and $deg(D) = 0$, then $D \sim 0$.

(Exercise 3) [2, Problems VI.3 C.] Let $deg(D)$ equal $2g-2$ and $dim\mathcal{L}(D) = g$. Show that $D$ is a canonical divisor.

## 13 Exercise sheet 13

The following are exercise $H$ and $I$ of Problems $VI.2$ in [2].

(Exercise 1) Let $X$ be a complex torus, and let $p$ be the zero of the group law on $X$. Let $z$ be a local cordinate on $X$ at $p$, and consider the Laurent tail divisor $Z = z^{-1}p$; note that $Z \in \mathcal{T}[0](X)$. Show that $Z$ is not in the image of $\alpha_0$, and conclude that $H^1(0) \neq 0$ for a complex torus.

(Exercise 2) Prove that $H^1(0) \neq 0$ for a complex torus $X$ via Riemann-Roch theorem.

(Exercise 3) Let $X$ be a complex torus. Fix a finite number of points $p_i$ on $X$, with local coordinate $z_i$ at $p_i$. Consider the Laurent tail divisor $Z = \sum_i c_i z_i^{-1} p_i$. Note that it is in $\mathcal{T}[0](X)$. Show that $Z = \alpha_0(f)$ for some global meromorphic function $f$ on $X$ iff $\sum\limits_i c_i = 0$. Prove the above with and without the help of Riemann-Roch theorem.

## References


[1] R. Cavalieri and E. Miles. "Riemann Surfaces and Algebraic Curves: A First Course in Hurwitz Theory". In: *Cambridge University Press* (2016).

[2] R. Miranda. *Algebraic Curves and Riemann Surfaces.* American Mathematical Society, 1995.