§ Mayer-Vietoris theorem

Def: a sequence of chain complexes

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$ is said to be

EXACT if $\operatorname{Im} f = \ker g$ i.e

$$\operatorname{Im} f_q = \ker g_q \quad \forall q \in \mathbb{Z}.$$

PROPOSITION: (Long exact sequence)

Let $0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ a short exact sequence of chain complexes.

Then there exist homomorphisms

$$\alpha_q : H_q(C'') \longrightarrow H_{q-1}(C') \text{ s.t.}$$

$$\cdots \xrightarrow{H_q(f)} H_q(C) \xrightarrow{H_q(g)} H_q(C'') \longrightarrow H_{q-1}(C') \to \cdots$$

is a long exact sequence.

PROOF: let's consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d, "\partial'_{q+1}"'] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d, "\partial_{q+1}"'] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"] & 0 \\
0 \arrow[r] & C'_q \arrow[r, hook, "f_q"] \arrow[d, "\partial'_q"'] & C_q \arrow[r, two heads, "g_q"] \arrow[d, "\partial_q"'] & C''_q \arrow[r] \arrow[d, "\partial''_q"] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d, "\partial'_{q-1}"'] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d, "\partial_{q-1}"'] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}$$

We want to define a homom.

$$\alpha_q : H_q(C'') \longrightarrow H_{q-1}(C')$$

$$\Psi$$

$[z''_q]$ where $z''_q \in Z''_q \subseteq C''_q$

g_q surjective $\forall q$ $\longrightarrow\!\!\!\!\rightarrow$

f_q injective $\forall q$ $\hookrightarrow$

LET US DEFINE α_q

$$
\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d, "\partial'_{q+1}"] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d, "\partial_{q+1}"] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"] & 0 \\
0 \arrow[r] & C'_q \arrow[r, hook, "f_q"] \arrow[d, "\partial'_q"] & C_q \arrow[r, two heads, "g_q"] \arrow[d, "\partial_q"] & C''_q \arrow[r] \arrow[d, "\partial''_q"] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d, "\partial'_{q-1}"] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d, "\partial_{q-1}"] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}
$$

$\exists\ c_q \mapsto z''_q$

$\exists\ c'_{q-1} \mapsto \partial_q(c_q)$

(1) g_q surjective $\exists\ c_q \in C_q : g_q(c_q) = z''_q$

(2) $g_{q-1}(\partial_q(c_q)) = \partial''_q(\underbrace{z''_q}_{\in \ker \partial''_q}) = 0$

$\Rightarrow \partial_q(c_q) \in \ker g_{q-1} = \mathrm{Im}(f_{q-1})$

3. $\exists\ c'_{q-1} \in C'_{q-1}$ s.t.

$$f_{q-1}(c'_{q-1}) = \partial_q(c_q)$$

Moreovere,

4. $$f_{q-2}(\partial'_{q-1}(c'_{q-1})) = \partial_{q-1}(\partial_q(c_q)) = 0$$

$\underset{f_{q-2} \text{ is injective}}{\Longrightarrow}$ $\partial'_{q-1}(c'_{q-1}) = 0 \Longrightarrow$ 5) $c'_{q-1} \in Z'_{q-1} \subset C'_{q-1}$

and we can consider $[c'_{q-1}]$ in $H_{q-1}(C')$.

Put $\alpha_q([z''_q]) = [c'_{q-1}]$

DEFINITION OF α_q

$$
\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d] & C''_{q+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{q} \arrow[r, hook, "f_{q}"] \arrow[d] & C_{q} \arrow[r, two heads, "g_{q}"] \arrow[d] & C''_{q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d] & C''_{q-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}
$$

LEFT TO CHECK:

1. α_q well defined
2. α_q homomorphism
3. $H_q(C^\bullet)$ is an exact sequence

1) ∂_q well defined

First, let us show that the definition does not depend on the choice of c_q, pre-image of z_q'' via g_q.

Let $\bar{c}_q$ be another pre-image of z_q'' via g_q. Then $g_q(\bar{c}_q - c_q) = 0$

$\Rightarrow \bar{c}_q - c_q \in \ker g_q = \operatorname{Im} f_q$

$\Rightarrow \exists\ \bar{c}'_q \in C'_q$ s.t.

$$\bar{c}_q = c_q + f_q(\bar{c}'_q) \Longrightarrow$$

$$\partial_q(\bar{c}_q) - \partial_q(c_q) = \partial_q(f_q(\bar{c}'_q)) \underset{\text{COMMUT. DIAGRAM}}{=} f_{q-1}(\partial'_{q-1}(\bar{c}'_q))$$

$\Rightarrow$ therefore the element in C_{q-1} defined starting from $\bar{c}_q$ differs from that defined from c_q by a boundary $\partial'_{q-1}(\bar{c}_q)$ and, therefore, such elements represent the same class in $H_q(C')$.

$$\bar{c}_q' \overset{②}{\longleftarrow} \bar{c}_q - c_q \overset{①}{\longmapsto} 0$$

$$\begin{tikzcd}
0 \arrow[r] & C_q' \arrow[r, hook, "f_q"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C_q \arrow[r, two heads, "g_q"] \arrow[d] & C_q'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{q-1}' \arrow[r, hook, "f_{q-1}"'] & C_{q-1} \arrow[r, two heads, "g_{q-1}"'] & C_{q-1}'' \arrow[r] & 0
\end{tikzcd}$$

③

$$\partial_q'(\bar{c}_q') \qquad \partial_q(\bar{c}_q) - \partial_q(c_q) \qquad ④$$

Let us show that the definition does not depend on the choice of the representative of the class $[z_q'']$. Let $\bar{z}_q''$ s.t. $[\bar{z}_q''] = [z_q'']$.

We have by definition that

$\bar{z}_q'' - z_q'' = \partial_{q+1}''(c_{q+1}'')$; since g_{q+1} is surjective $\exists\ c_{q+1} \in C_{q+1}$ s.t.

$g_{q+1}(c_{q+1}) = c''_{q+1}$

So $\bar{z}''_q - z''_q = \partial''_{q+1}\left(g_{q+1}(c_{q+1})\right)$
$$= g_q\left(\partial_{q+1}(c_{q+1})\right)$$

Now, since $\partial_q\left(\partial_{q+1}(c_{q+1})\right) = 0$ it follows that the definition of α_q does not depend on the chosen representative.

c_{q+1} $\quad$ c''_{q+1}

$$\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d, "\partial'_{q+1}"] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d, "\partial_{q+1}"] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"] & 0 \\
0 \arrow[r] & C'_q \arrow[r, hook, "f_q"] \arrow[d, "\partial'_q"] & C_q \arrow[r, two heads, "g_q"] \arrow[d, "\partial_q"] & C''_q \arrow[r] \arrow[d, "\partial''_q"] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d, "\partial'_{q-1}"] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d, "\partial_{q-1}"] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}$$

$z''_q - z_q$ $\quad$ 0

② α_q homomorphism

Recall that f_q, g_q are homomorphisms.
Let $[z''_q]$, $[\bar{z}''_q]$ two classes in $H_q(C''_q)$. Consider $\alpha_q([z''_q]) = [c'_{q-1}]$

$$\alpha_q([\bar{z}''_q]) = [\bar{c}'_{q-1}]$$

Now we want to show that

$$\alpha_q([z''_q + \bar{z}''_q]) = [c'_{q-1} + \bar{c}'_{q-1}]$$

① via g_q (surjective homomorphism) choose $c_q + \bar{c}_q$ as preimage of $z''_q + \bar{z}''_q$

② Now, consider $\partial_q(c_q + \bar{c}_q) = \partial_q(c_q) + \partial_q(\bar{c}_q)$ and take the unique preimage (f_{q-1} injective) via f_{q-1}, which must be $c'_{q-1} + \bar{c}'_{q-1}$ $\Longrightarrow$

$$\alpha_q([\bar{z}_q'' + z_q'']) = [c_{q-1}' + \bar{c}_{q-1}'] = \alpha_q([z_q'']) + \alpha_q([\bar{z}_q''])$$

③ $H_q(C^\bullet)$ is an exact sequence

$$\cdots \xrightarrow{H_q(f)} H_q(C) \xrightarrow{H_q(g)} H_q(C'') \xrightarrow{\alpha_q} H_{q-1}(C') \to \cdots$$

we show together the exactness in $H_q(C'')$.

$$\boxed{\operatorname{Im} H_q(g) \subseteq \ker \alpha_q}$$:

c_q q-cycle $\quad H_q(g)([c_q]) = [z_q'']$

$\exists\, c_{q+1}''$ s.t. $g_q(c_q) = z_q'' + \partial_{q+1}''(c_{q+1}'')$

by definition of g_q, one can take c_q as preimage of z_q'' via g_q and $\partial_q(c_q) = 0$ because c_q q-cycle

$\Rightarrow$ the preimage via f_{q-1} give us what we want!

$$\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d, "\partial'_{q+1}"] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d, "\partial_{q+1}"] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"] & 0 \\
0 \arrow[r] & C'_q \arrow[r, hook, "f_q"] \arrow[d, "\partial'_q"] & C_q \arrow[r, two heads, "g_q"] \arrow[d, "\partial_q"] & C''_q \arrow[r] \arrow[d, "\partial''_q"] & 0 \\
0 \arrow[r] & {[0]}\ C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d, "\partial'_{q-1}"] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d, "\partial_{q-1}"] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}$$

$$\boxed{\operatorname{Ker} \alpha_q \subseteq \operatorname{Im} H_q(g)}\ ;$$

$$\cdots \xrightarrow{H_q(f)} H_q(C) \xrightarrow{H_q(g)} H_q(C'') \xrightarrow{\alpha_q} H_{q-1}(C') \to \cdots$$

$[z''_q] \in H_q(C'')$ s.t. $\alpha_q([z''_q]) = [0]$

z''_q q-cycle $\rightsquigarrow C_q$ via g_q $\rightsquigarrow C'_{q-1}$ via f_{q-1}

$\Rightarrow \alpha_q([z''_q]) = [\partial'_q(c'_q)]$ $\quad [c'_{q-1}]$

take $c_q - f_q(c'_q)$ as candidate as preimage of z''_q via g_q

$$\partial_q(c_q) - \partial_q f_q(c'_q)$$

$$= \partial_q(c_q) - f_{q-1}(\partial'_q(c'_q))$$

$$= \partial_q(c_q) - f_{q-1}(c'_{q-1}) = 0$$

COMMUTATIVE DIAGRAM

$\Longrightarrow$ $c_q - f_q(c'_q)$ is a q-cycle and therefore

$$[c_q - f_q(c'_q)] \in H_q(C)$$

In conclusion

$$H_q(g)([c_q - f_q(c'_q)]) = [g_q(c_q)] = [z''_q]$$

$\operatorname{Im} f_q = \operatorname{Ker} g_q$

EXERCISE: show the exactness in $H_q(C')$ and in $H_q(C)$ $\square$

<u>Def</u>: $C = \{C_q, \partial_q\}$ $C' = \{C'_q, \partial'_q\}$
two chain complexes
direct sum of C and C'
$$C \oplus C' = \{C_q \oplus C'_q, \partial_q \oplus \partial'_q\}$$
$$\partial_q \oplus \partial_q' : C_q \oplus C_q' \to C_{q-1} \oplus C'_{q-1}$$
$$(c, c') \mapsto (\partial_q(c), \partial'_q(c'))$$

X topological space, $X_1, X_2 \subset X$ subspace of X

$$\begin{tikzcd} & X_1 \arrow[dr, "i_1"] & \\ X_1 \cap X_2 \arrow[ur, "j_1"] \arrow[dr, "j_2"'] \arrow[rr, phantom, "\circlearrowright"] & & X \\ & X_2 \arrow[ur, "i_2"'] & \end{tikzcd}$$

induces

$$\begin{tikzcd} & S^A(X_1) \arrow[dr, "S(i_1)"] & \\ S^A(X_1 \cap X_2) \arrow[ur, "S(j_1)"] \arrow[dr, "S(j_2)"'] \arrow[rr, phantom, "\circlearrowright"] & & S^A(X) \\ & S^A(X_2) \arrow[ur, "S(i_2)"'] & \end{tikzcd}$$

One can consider the following exact short sequences

$$0 \to S_q(X_1 \cap X_2) \xrightarrow{\varphi_q} S_q(X_1) \oplus S_q(X_2) \xrightarrow{\psi_q} \underbrace{\langle S_q(X_1), S_q(X_2) \rangle}_{\text{A-module subgroup of } S_q(X) \text{ generated by } S_q(X_1) \text{ and } S_q(X_2)} \to 0$$

$\varphi_q : \left(S_q(J_1), -S_q(J_2) \right)$

$\psi_q : \quad S_q(i_1) + S_q(i_2)$

$\{\varphi_q\} = \varphi \qquad \{\psi_q\} = \psi$ chain complexes homomorphisms

$$\underbrace{0 \to S(X_1 \cap X_2) \xrightarrow{\varphi} S(X_1) \oplus S(X_2) \xrightarrow{\psi} S\{X_1, X_2\} \to 0}_{}$$

short exact sequence

$S\{X_1, X_2\} = \{\langle S_q(X_1), S_q(X_2) \rangle, \partial_q\}$

$\Longrightarrow \exists$ Long exact sequence in hology

PROPOSITION : (Long exact sequence)

what about $S\{X_1, X_2\}$?

Mayer-Vietoris theorem

If $X = \mathring{X}_1 \cup \mathring{X}_2$ ← open then the following is a long exact sequence

$$\cdots \to H_q(X_1 \cap X_2) \xrightarrow{\Phi_q} H_q(X_1) \oplus H_q(X_2) \xrightarrow{\Psi_q} H_q(X) \xrightarrow{\alpha_q} H_{q-1}(X_1 \cap X_2) \to \cdots$$

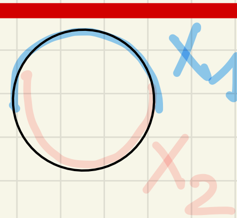
where $\Phi_q = \left(H_q(j_1), -H_q(j_2)\right)$

$\Psi_q = H_q(i_1) + H_q(i_2)$

α_q as in PROPOSITION : (Long exact sequence)

Applications:

- $X = S^1$



$A = \mathbb{Z}$

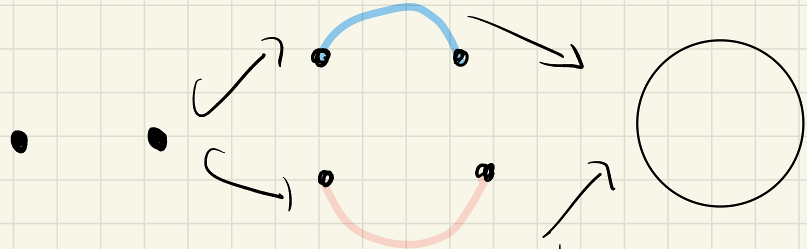
$\boxed{H_i(S^1; \mathbb{Z})}$

$X_1 \sim \mathbb{R}$
$X_2 \sim \mathbb{R}$

$X_1 \cap X_2 \sim pt \sqcup pt$

$$0 \to H_1(S^1; \mathbb{Z}) \xrightarrow{\alpha_1} H_0(pt) \oplus H_0(pt)$$

$$\xrightarrow{\Phi_0} \underbrace{H_0(\mathbb{R})}_{\mathbb{Z}} \oplus \underbrace{H_0(\mathbb{R})}_{\mathbb{Z}} \xrightarrow{\Psi_0} H_0(S^1) \to 0$$



$$H_0(pt)\oplus H_0(pt) \xrightarrow{\phi_0} H_0(\mathbb{R})\oplus H_0(\mathbb{R})$$

$$(1,0) \longmapsto (1,-1)$$

$$(0,1) \longmapsto (1,-1)$$

1) $H_0(S^1) \cong \dfrac{H_0(\mathbb{R})\oplus H_0(\mathbb{R})}{\ker \psi_0} = \dfrac{H_0(\mathbb{R})\oplus H_0(\mathbb{R})}{\operatorname{Im}\phi_0}$

2) $H_1(S^1) \simeq \operatorname{Im}\alpha_1 = \ker\phi_0$

Injective

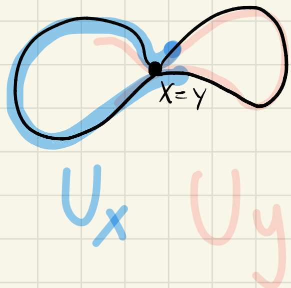
$$\operatorname{Im}\phi_0 \simeq \mathbb{Z} \implies \ker\phi_0 \simeq \mathbb{Z}$$

$$\implies H_0(S^1) \simeq \mathbb{Z}\oplus\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}$$

$$H_1(S^1) \simeq \mathbb{Z}$$

• Fix $x \in X$, Fix $y \in Y$ $\quad X, Y$ topological spaces

$X \vee Y = X \sqcup Y \big/ {x=y}$



$A = \mathbb{Z}$

$$\boxed{H_i(X \vee Y; \mathbb{Z})}$$

$U_x \cap U_y \sim pt$

$U_x \sim X$

$U_y \sim Y$

$$\cdots \to \underbrace{\cancel{H_q(pt)}}_{0} \to H_q(X) \oplus H_q(Y) \xrightarrow{\cong} H_q(X \vee Y) \to \underbrace{\cancel{H_{q-1}(pt)}}_{0} \to \cdots$$

$q > 1$

INJECTIVE

$$\cdots \to \underbrace{\cancel{H_1(pt)}}_{0} \xrightarrow{\phi_1} H_1(X) \oplus H_1(Y) \xhookrightarrow{\psi_1} H_1(X \vee Y) \xrightarrow{\alpha_1} H_0(pt) \to \cdots$$

$$\xrightarrow{\phi_0} H_0(X) \oplus H_0(Y) \xrightarrow{\psi_0} H_0(X \vee Y) \to 0$$

$$H_0(X \vee Y) \simeq H_0(X) \oplus H_0(Y) \big/ \ker \psi_0 = H_0(X) \oplus H_0(Y) \big/ \mathrm{Im}\, \phi_0$$

$\mathrm{Im}\, \phi_0 \simeq \mathbb{Z}$

$\Rightarrow \ker \phi_0 \simeq 0 \Rightarrow \alpha_1 = 0$ (exactness)

$\Rightarrow \psi_1$ isomorphism

$$H_q(X \vee Y) = \begin{cases} H_q(X) \oplus H_q(Y) & q \neq 0 \\ \dfrac{H_0(X) \oplus H_0(Y)}{\mathbb{Z}} & q = 0 \end{cases}$$

EXERCISE:

1. $X = S^n$ n-sphere $\qquad A = \mathbb{Z}$

Prove that
$$H_q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise} \end{cases}$$

CASE 0: $S^0 \sim pt \sqcup pt$

CASE 1: SEEN!

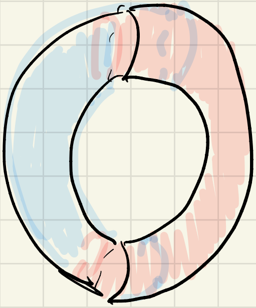
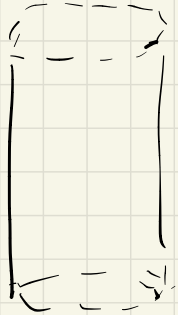
2. $X = T$ torus

Prove that
$$H_q(T) = \begin{cases} \mathbb{Z} & q = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

3. $X = T_g = \#_g T$ torus of genus g

Prove that
$$H_q(X) = \begin{cases} \mathbb{Z} & q = 0, 2 \\ \bigoplus_{2g} \mathbb{Z} & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

2

X  

two open cylinders

$S^1 \times \mathbb{R}^2 \sim S^1$

$X_1 \ , \ X_2$

$X_1 \cap X_2 \sim S^1 \sqcup S^1$

$$H_q(S^1) \oplus H_q(S^1) \to H_q(X_1) \oplus H_q(X_2) \to H_q(X) \to H_{q-1}(S^1 \sqcup S^1)$$

$$H_q(S^1) = \begin{cases} \mathbb{Z} & q = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$q > 1 \implies H_q(X) = 0$

$q = 1$

$$0 \to H_2(X) \xhookrightarrow{\alpha_2} H_1(S^1) \oplus H_1(S^1) \xrightarrow{\phi_1}$$

$$H_1(X_1) \oplus H_1(X_2) \xrightarrow{\psi_1} H_1(X) \xrightarrow{\alpha_1}$$

$$H_0(S^1) \oplus H_0(S_1) \xrightarrow{\phi_0} H_0(X_1) \oplus H_0(X_2)$$

$$\xrightarrow{\psi_0} H_0(X) \longrightarrow 0$$

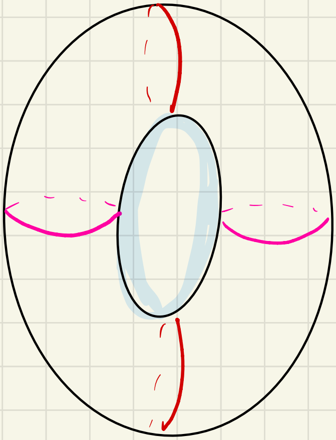
- $H_0(X) \simeq \mathbb{Z} \implies \ker \phi_0 = \mathbb{Z}$
- $\operatorname{Im} \alpha_i = \ker \phi_{i-1}$ for $i = 2, 1$ exact sequence
- $\operatorname{Im} \phi_1 = \mathbb{Z} \implies \ker \phi_1 = \mathbb{Z}$

use definition ϕ_1 use $H_1(S^1) \oplus H_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$

- α_2 is injective : $\operatorname{Im} \alpha_2 \simeq \underbrace{H_2(X)}_{\mathbb{Z}} = \ker \phi_1$
- $\operatorname{Im} \phi_1 = \mathbb{Z} \implies \ker \psi_1 = \mathbb{Z}$
$\implies \operatorname{Im} \psi_1 = \mathbb{Z}$

- $H_1(X) \simeq \underbrace{\ker \alpha_1}_{\underbrace{\operatorname{Im} \psi_1}_{\mathbb{Z}}} \oplus \underbrace{\operatorname{Im} \alpha_1}_{\underbrace{\ker \phi_0}_{\mathbb{Z}}} \simeq \mathbb{Z} \oplus \mathbb{Z}$

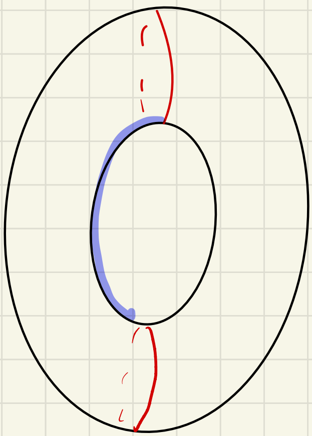
$\operatorname{Im} \Psi_1$



generators of respectively $\underbrace{H_1(X_1) \text{ and } H_1(X_2)}$

when we glue this becomes the same class in homology

$X_1 \cap X_2 \sim S^1 \sqcup S^1$



$\Big[$  $\Big] = [0]$ in $H_0(X_1)$

But the sum gives a class in $H_1(X)$