Homology of $\mathbb{P}^2(\mathbb{C})$, $\mathbb{P}^2(\mathbb{R})$

$A = \mathbb{Z}$

$K = \mathbb{C}, \mathbb{R}$

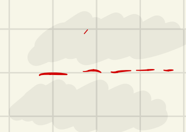
$\mathbb{P}^2(K) = \bigcup_3 K^2 \qquad K^2 \cap K^2 = K^*$

we knew that

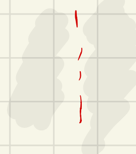
$\boxed{\mathbb{R}^n \text{ or } \mathbb{R}^{2n} \simeq \mathbb{C}^n}$

- $H_i(K^n) = H_i(pt)$
- $\mathbb{R}^* \underset{\text{homeo}}{\cong} \mathbb{R} \sqcup \mathbb{R} \qquad H_i(\mathbb{R}^*) \simeq H_i(\mathbb{R}) \oplus H_i(\mathbb{R}) \quad \forall i$
- $\mathbb{R}^2 \setminus \{pt\} = X \simeq \mathbb{C}^*$

$X_1 = \mathbb{R}^2 \setminus \{x=0\} \overset{\text{homeo}}{\simeq} \mathbb{R}^2 \sqcup \mathbb{R}^2$



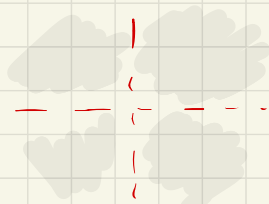
$X_2 = \mathbb{R}^2 \setminus \{y=0\} \overset{\text{homeo}}{\simeq} \mathbb{R}^2 \sqcup \mathbb{R}^2$



$X_1 \cup X_2 = X$

$X_1 \cap X_2 = \mathbb{R}^2 \setminus (\{y=0\} \cup \{x=0\})$

$\overset{\text{homeo}}{\simeq} \bigsqcup_4 \mathbb{R}^2$



$$\cdots H_q(X_1 \cap X_2) \to H_q(X_1) \oplus H_q(X_2) \to H_q(X) \to H_{q-1}(X_1 \cap X_2)$$

$$0 \to H_1(X) \hookrightarrow \underbrace{\bigoplus_4 H_0(\mathbb{R}^2)}_{A^4} \xrightarrow{\varphi} \underbrace{\bigoplus_2 H_0(\mathbb{R}^2)}_{A^2 \oplus A^2}$$

$$\oplus \bigoplus_2 H_0(\mathbb{R}^2) \twoheadrightarrow \underbrace{H_0(X)}_{A} \to 0$$

$$(1,0,0,0) \hookrightarrow \begin{array}{cc} 1 & \\ 0 & \\ -1 & 0 \end{array}$$

example

$$(1,0,0,0) \longmapsto (1,0,-1,0)$$

$$\det\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} = 0 \qquad \mathrm{rk} = 3$$

$$\Longrightarrow H_1(X) = \ker\varphi \simeq A$$

$$\Rightarrow \; H_i(\mathbb{C}^*) \begin{cases} H_1(\mathbb{R}^2 \setminus \{pt\}) \simeq A \\ H_0(\mathbb{R}^2 \setminus \{pt\}) \simeq A \\ H_i(\mathbb{R}^2 \setminus \{pt\}) = 0 \quad \forall i > 1 \end{cases}$$

$$\underbrace{\mathbb{P}^2(\mathbb{C})}_{Z} = \{x \neq 0\} \cup \{y \neq 0\} \cup \{z \neq 0\}$$

$$U_1 \cup U_2 \cup U_3$$

$$= \underbrace{\bigcup_{i=1}^{2} U_i}_{Y} \cup \underbrace{U_3}_{\mathbb{C}^2}$$

$$U_i \simeq \mathbb{C}^2$$

$$U_i \cap U_j \simeq \mathbb{C}^*$$

$$\bigcap_{i=1}^{3} U_i \simeq (\mathbb{C}^*)^2$$

exercise: $(\mathbb{C}^*)^2 \sim S^3$ $\quad H_i = \begin{cases} 0 & \text{otherwise} \\ A & i = 0, 3 \end{cases}$

Let us compute the homology of Y:

$$Y = \underset{X_1}{\underset{\shortparallel}{\mathbb{C}^2}} \cup \underset{X_2}{\underset{\shortparallel}{\mathbb{C}^2}} \quad \text{s.t.} \quad X_1 \cap X_2 \sim \mathbb{C}^*$$

$$0 \to H_2(Y) \xrightarrow{\cong} \overset{\cong A}{H_1(\mathbb{C}^*)} \to \cancel{H_1(\mathbb{C}^2) \oplus H_1(\mathbb{C}^2)} = 0$$

$$\to \underset{=0}{\cancel{H_1(Y)}} \to \underset{\cong A}{H_0(\mathbb{C}^*)} \to \underset{A \oplus A}{H_0(\mathbb{C}^2) \oplus H_0(\mathbb{C}^2)}$$

$$\twoheadrightarrow \underset{\underset{A}{\cong}}{H_0(Y)} \to 0$$

$$H_i(Y) = \begin{cases} A & i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Homology of $Z = \mathbb{P}^2(\mathbb{C})$

$X_1 = Y \qquad X_2 = \mathbb{C}^2$

$X_1 \cap X_2 = (\mathbb{C}^*)^2$ $\quad A$

$0 \to H_4(Z) \xrightarrow{\simeq} H_3((\mathbb{C}^*)^2) \to 0$

$0 \to H_3(Z) \to 0$

$$0 \to \overset{A}{H_2(Y)} \oplus \overset{0}{\cancel{H_2(\mathbb{C}^2)}} \xrightarrow{\simeq}$$

$$H_2(Z) \to H_1((\mathbb{C}^*)^2) \to 0$$

$$0 \to \underset{0}{H_1(Y)} \oplus \underset{0}{H_1(\mathbb{C}^2)} \to$$

$$\underset{0}{H_1(Z)} \hookrightarrow H_0(\underset{A}{(\mathbb{C}^*)^2})$$

$$\to \underset{A}{H_0(Y)} \underset{\oplus}{\oplus} \underset{A}{H_0(\mathbb{C}^2)} \to \underset{A}{H_0(Z)} \to 0$$

$$\Rightarrow \quad H_i(\mathbb{P}^2(\mathbb{C})) = \begin{cases} A & i = 0 \bmod 2,\ 0 \leq i \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

__ex:__ $\mathbb{P}^1(\mathbb{C}) \simeq S^2$

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C}_{x_0 \neq 0} \cup \mathbb{C}_{x_1 \neq 0} \quad \text{s.t.}$$

$$\mathbb{C} \cap \mathbb{C} = \mathbb{C}^*$$

$$H_q(\mathbb{C}^*) = \begin{cases} A & i = 0,1 \\ 0 & \text{o/w} \end{cases}$$

$$0 \to H_2(\mathbb{P}^1(\mathbb{C})) \xrightarrow{\sim} H_1(\mathbb{C}^*) \to 0$$

$$0 \to \cancel{H_1(\mathbb{P}^1(\mathbb{C}))} \xrightarrow{0} H_0(\mathbb{C}^*) \hookrightarrow H_0(\mathbb{C}) \oplus H_0(\mathbb{C})$$

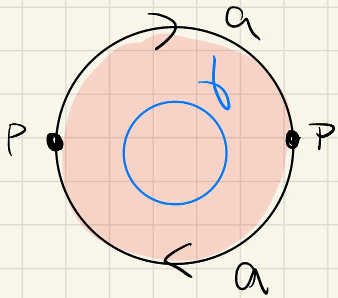
$$\to H_0(\mathbb{P}^1(\mathbb{C})) \to 0$$

__exercise__: compute homology of $\mathbb{P}^2(\mathbb{R})$ $\quad A = \mathbb{Z}$ and

$A = \mathbb{Z}/2\mathbb{Z}$

$$\cdots H_q(X_1 \cap X_2) \xrightarrow{\varphi_q} H_q(X_1) \oplus H_q(X_2) \xrightarrow{\phi_q} H_q(X) \xrightarrow{\alpha_q} H_{q-1}(X_1 \cap X_2)$$

Exercise :



$A = \mathbb{Z}$

$\mathbb{RP}^2 = X = S^2 /$ antipodal relation

a ∼ X_1 (open Mobius strip)

• ∼ X_2 (disk)

$X_1 \cap X_2 \sim S^1 = \gamma$

$q = 1$ $\quad \mathbb{Z}\langle\gamma\rangle \xrightarrow{\varphi_1} 0 \oplus \mathbb{Z}\langle a\rangle$

$[\gamma] \longmapsto (0, [2a])$

φ_1 is injective

$\Rightarrow H_2(\mathbb{RP}^2; \mathbb{Z}) = 0$

$H_1(\mathbb{RP}^2; \mathbb{Z}) \simeq \mathbb{Z}\langle a\rangle / \ker\varphi_1 \simeq \mathbb{Z}\langle a\rangle / \mathrm{Im}\,\varphi_1$

$\simeq \mathbb{Z}\langle a\rangle / \mathbb{Z}\langle 2a\rangle \simeq \mathbb{Z}/2\mathbb{Z}$

Now $\boxed{A = \mathbb{Z}/2\mathbb{Z}}$ $X = \mathbb{R}P^2$

φ_1 is trivial $\Rightarrow H_2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

$$H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\langle a \rangle \big/ \operatorname{Im} \varphi_1$$

$$\simeq \mathbb{Z}/2\mathbb{Z}$$

Homology of a pair

$B \subset X$ $\quad (X, B)$ pair of top. space

sub. top. space $\quad$ top space

$S(X) = \{S_q(X), \partial_q\}$

$S(B) = \{S_q(B), \partial_q\}$

$i : B \hookrightarrow X$ induces $\forall q$

$S_q(i) : S_q(B) \longrightarrow S_q(X)$:

$$S_q(i)\left(\sum_{j=1}^{k} n_j \sigma_j\right) = \sum_{j=1}^{k} m_j (i \circ \sigma_j)$$

$$\boxed{\Delta_q \xrightarrow{\sigma_q} B \xrightarrow{i} X}$$

$$\forall q \quad \boxed{S_q(X, B) = S_q(X) / S_q(B)}$$

relative q-chains of X modulo B

one ~~con~~ can construct

$\bar{\partial}_q : S_q(X, B) \longrightarrow S_{q-1}(X, B)$

s.t. $\{S_q(X, B), \bar{\partial}_q\}$ is a chain complex $\boxed{\overline{\partial_q}([s]) := [\partial_q(s)]}$

$$\begin{tikzcd}
0 \arrow[r] & S_q(B) \arrow[r, "S_q(i)"] \arrow[d, "\partial_q"] & S_q(X) \arrow[r, "\pi_q"] \arrow[d, "\partial_q"] & S_q(X,B) \arrow[r] \arrow[d, dashed, "\overline{\partial_q}"] & 0 \\
0 \arrow[r] & S_{q-1}(B) \arrow[r, "S_{q-1}(i)"] & S_{q-1}(X) \arrow[r, "\pi_{q-1}"] & S_{q-1}(X,B) \arrow[r] & 0
\end{tikzcd}$$

well defined, in fact

if $s' = s + s_B \qquad s_B \in S_q(B)$

$\Rightarrow \partial_q(s_B) \in S_{q-1}(s_B)$

$\Rightarrow [\partial_q(s')] = [\partial_q(s) + \partial_q(s_B)]$

$= [\partial_q(s)]$

$S(X,B) = \{ S_q(X,B), \overline{\partial_q} \}$

RELATIVE SINGULAR COMPLEX

OR SINGULAR COMPLEX OF THE PAIR (X, B).

$$\boxed{H_q(S(X,B)) := H_q(X,B)}$$

Relative homology or homology of the pair (X, B).

Def: relative q-cycles of X mod B

$$\boxed{Z_q(X,B)} = \{ s \in S_q(X) ; \partial_q s \in S_{q-1}(B) \}$$

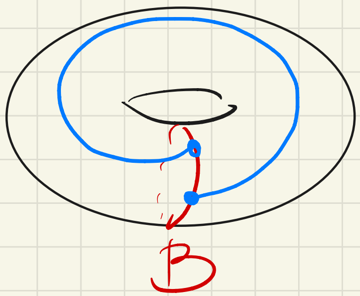
relative q-boundaries of X mod B

$$\boxed{B_q(X,B)} = \{ s \in S_q(X) : s = \partial_{q+1}(s') + s_B \quad s' \in S_{q+1}(X) \; s_B \in S_q(B) \}$$

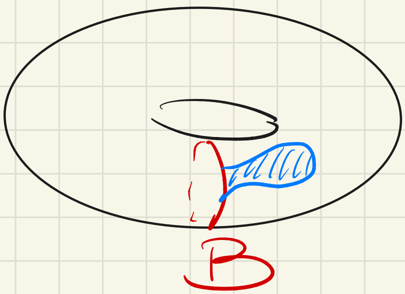
Rk: • $Z_q(X) \subseteq Z_q(X,B)$

• $B_q(X) \subseteq B_q(X,B)$

Example: $X = T$ $\quad B = S^1$



relative 1-cycle
(not a cycle)



relative 1-boundary
(not a boundary)

• $B_q(X,B) \subseteq Z_q(X,B)$ by def.

$\Rightarrow$ can consider $Z_q(X,B) / B_q(X,B)$

Theorem: $H_q(X,B) \simeq \dfrac{Z_q(X,B)}{B_q(X,B)} \quad \forall q$

Proof: $H_q(X,B) = \dfrac{\ker \bar{\partial}_q}{\operatorname{Im} \bar{\partial}_{q+1}}$

- $\pi_q : \underset{\underset{Z_q(X,B),\, B_q(X,B)}{\cup|}}{S_q(X)} \longrightarrow \dfrac{S_q(X)}{S_q(B)} =: S_q(X,B)$

- $\bar{\partial}_q : S_q(X,B) \longrightarrow S_{q-1}(X,B)$

(1) $\pi_q(Z_q(X,B)) = \ker(\bar{\partial}_q)$

$\boxed{\subseteq}$ $s \in Z_q(X,B)$, then $\bar{\partial}_q[s] = [\overset{S_{q-1}(B)}{\overset{\cup}{\partial_q(s)}}] \overset{!}{=} [0]$

$\boxed{\supseteq}$ if $\bar{\partial}_q[s] = 0$, then

$\partial_q(s) \in S_{q-1}(B)$.

(2) $\ker\left(\pi_q|_{Z_q(X,B)} : Z_q(X,B) \longrightarrow \ker(\bar{\partial}_q)\right)$

$$= \{ s \in Z_q(X,B) : [s] = 0 \text{ , i.e. } s \in S_q(B) \}$$

$$\Longrightarrow \ker \bar{\partial}_q \cong \frac{Z_q(X,B)}{S_q(B)}$$

1ST ISOMORPH. THEOREM

3) $\Pi_q(B_q(X,B)) = \operatorname{Im}(\bar{\partial}_{q+1})$.

$\subseteq$ if $s \in B_q(X,B)$, then $[s] \in \operatorname{Im} \bar{\partial}_{q+1}$

In fact $s = \partial_{q+1}(s') + s_B$; then

$$[s] = [\partial_{q+1}(s')] = \bar{\partial}_{q+1}(s').$$

$\supseteq$ if $[s] = \bar{\partial}_{q+1}[s'] \Longrightarrow$

$$[s] = [\bar{\partial}_{q+1}(s')] \text{ , i.e.}$$

$$s - \bar{\partial}_{q+1}(s') \in S_q(B).$$

$$\ker \left(\Pi_q \big|_{B_q(X,B)} : B_q(X,B) \longrightarrow \operatorname{Im}(\bar{\partial}_q) \right)$$

$$= \{ s \in B_q(X,B) \; ; \; [s] = 0 \} = S_q(B)$$

$$\underset{\text{1ST ISOM. TH}}{\Longrightarrow} \operatorname{Im}(\bar{\partial}_{q+1}) \simeq B_q(X,B) / S_q(B)$$

$\square$ (3rd isom th)

we get a short exact sequence of chain complexes.

$$0 \to S(B) \xrightarrow{S(i)} S(X) \xrightarrow{S(\pi)} S(X,B) \to 0$$

inducing a long exact sequence in homology

$$\boxed{\cdots H_q(B) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(\pi)} H_q(X,B) \xrightarrow{\partial_q} H_{q-1}(B)}$$

LONG EXACT SEQUENCE OF THE PAIR (X,B)

<u>Prop/ex</u>: $H_0(X,B)$ free A-module generated by the connected components not intersecting B.

ex: D^n disk S^{n-1} $(n-1)$-sphere $\overset{A}{=} \mathbb{Z}$

Compute $H_q(D^n, S^{n-1})$ $\forall q$

$$H_q(D^n, S^{n-1}) \simeq \begin{cases} \{0\} & q \neq n \\ \mathbb{Z} & q = n \end{cases} \quad \forall q, n$$

STEPS:

$q \geq 2$ $\quad H_q(D^n, S^{n-1}) \simeq \begin{cases} \{0\} & q \neq n \\ \mathbb{Z} & q = n \end{cases}$

if $n \geq 2$ $\quad H_1(D^n, S^{n-1}) = H_0(D^n, S^{n-1}) = 0$

if $n = 1$ $\quad H_1(D^1, S^0) = \mathbb{Z}$

$H_0(D^1, S^0) = 0$

Proposition (For those knowing deformation retraction)
Fact (otherwise)

Let $B \subset X$ be a retract, then

$$H_q(X) \simeq H_q(B) \oplus H_q(X, B)$$

Application: $k, n \in \mathbb{Z}_{\geq 0}$ $\underbrace{S^n}_{\text{unit sphere}} \subset \mathbb{R}^{n+1}$

If $k < n$ then S^k is not a retract of S^n.

Pf: If it were the case $q = k$

$$0 \cong H_k(S^n) \cong \underbrace{H_k(S^k)}_{\mathbb{Z}} \oplus H_k(S^n, S^k)$$

Excision

Def: $f : (X, B) \longrightarrow (Y, C)$ is a pair morphism if $f : X \longrightarrow Y$ continuous s.t. $f(B) \subset C$

Given a pair morphism

$$f : (X, B) \longrightarrow (Y, C)$$

can consider

$$Sq(f): Sq(X) \longrightarrow Sq(Y)$$

$$Sq(f): Sq(B) \longrightarrow Sq(C)$$

and, therefore

$$\overline{Sq}(f): Sq(X,B) \longrightarrow Sq(Y,C)$$

$\{\overline{Sq}(f)\}$ turn out to be chain complex morphisms inducing $Hq(f): Hq(X,B) \to Hq(Y,C)$

PARTICOLAR CASE

(X,B) be a pair of spaces

$W \subset B$

$$i: (X \setminus W, B \setminus W) \hookrightarrow (X,B)$$

$$Hq(i): Hq(X \setminus W, B \setminus W) \longrightarrow Hq(X,B)$$

<u>Def</u>: $i: (X \setminus W \; B \setminus W) \hookrightarrow (X,B)$ is an EXCISION if $Hq(i)$ is an isomorphism $\forall q$.

Theorem (EXCISION):

(X, B) a pair of spaces

$W \subset B$ s.t. $\overline{W} \subset \mathring{A}$ ← interior / open, then

W can be EXCISED

Application (1):

let X be a top. space s.t. every point is a closed set.

V be a neigh. of $x \in X$

Then $i : (V, V \setminus \{x\}) \hookrightarrow (X, X \setminus \{x\})$

is an excision $\forall q \in \mathbb{Z}$.

Proof (1): we want to use the theorem

$$\underbrace{\overline{X \setminus V}}_{W} = X \setminus \mathring{V} \subset X \setminus \{x\} = \underbrace{(X \setminus \{x\})^{\circ}}_{\mathring{B}}$$

$\Rightarrow X \setminus V$ can be excised

from the pair $(X, X \setminus \{x\})$

and $H_q(V, V \setminus \{x\}) \cong H_q(X, X \setminus \{x\}) \ \forall q$

2 $H_q(V, V\setminus\{0\})$ $0\in V\subset \mathbb{R}^n$?

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & D^n \arrow[d, hook] \\ \mathbb{R}^n\setminus\{0\} \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

$i: (D^n, S^{n-1}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$

induces

$$\begin{tikzcd}[column sep=small] H_q(S^{n-1}) \arrow[r] \arrow[d] & H_q(D^n) \arrow[r] \arrow[d] & H_q(D^n, S^{n-1}) \arrow[r] \arrow[d] & H_{q-1}(S^{n-1}) \arrow[r] \arrow[d] & H_{q-1}(D^n) \arrow[d] \\ H_q(\mathbb{R}^n\setminus\{0\}) \arrow[r] & H_q(\mathbb{R}^n) \arrow[r] & H_q(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[r] & H_{q-1}(\mathbb{R}^n\setminus\{0\}) \arrow[r] & H_{q-1}(\mathbb{R}^n) \end{tikzcd}$$

$\rightarrow$ are isomorphisms because

S^{n-1} is a deformation retract of $\mathbb{R}^n\setminus\{0\}$

D^n " " " " of $\mathbb{R}^n$

$\Longrightarrow$ $H_q(D^n, S^{n-1}) \simeq H_q(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$

FIVE LEMMA (exercise) (2 exact rows + 4 external vertical isom $\Longrightarrow$ central isom)

$$H_q(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{Z} & q = n \\ \{0\} & q \neq n \end{cases}$$

COROLLARY: (DIMENSION INVARIANCE)

Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ s.t. $\underset{\text{homeomorphic}}{U \cong V}$. Then $n = m$.

Proof: $\varphi : \underset{x}{U} \longrightarrow \underset{y = \varphi(x)}{V}$ homeomorphism

$$\begin{aligned}
\mathbb{Z} &\overset{2}{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \\
&\cong H_n(U, U \setminus \{x\}) \\
&\cong H_n(V, V \setminus \{\varphi(x)\}) \\
&\cong H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{\varphi(x)\})
\end{aligned}$$

$\square$

Poincaré duality

- Let X be a compact orientable connected manifold

$\dim X = n$. Let $Y \subset X$ a compact connected orientable topological subvariety of dim m, then

$$i : Y \hookrightarrow X \text{ induces}$$

$$i_* : H_m(Y) \longrightarrow H_m(X)$$
$$\mathbb{Z} \cong H_m(Y), \quad 1 \longmapsto [Y]$$

HOMOLOGY CLASS REALISED BY Y in X.

Let X be a compact manifold of dimension m.

① if X is orientable then there exists a perfect pairing

$$\boxed{H_k(X;\mathbb{Z}) \times H_{m-k}(X;\mathbb{Z}) \longrightarrow \mathbb{Z}}.$$

In particular $H_k(X;\mathbb{Z}) \simeq (H_{m-k}(X,\mathbb{Z}))^{\vee}$

② if X is not orientable then there exists a perfect pairing

$$H_k(X;\mathbb{Z}/2\mathbb{Z}) \times H_{m-k}(X;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

In particular $H_k(X;\mathbb{Z}/2\mathbb{Z}) \simeq (H_{m-k}(X,\mathbb{Z}/2\mathbb{Z}))^{\vee}$

where $(H_r(X;A))^{\vee} = \operatorname{Hom}(H_r(X;A),A)$

(pairing A-bilinear map $M \times N \to L$
$\iff \phi : M \to \operatorname{Hom}_A(N,L)$
perfect ϕ isomorphism, where M,N,L A-modules)

COROLLARY: $(H_0(X;A))^{\vee} \simeq H_m(X;A)$

example: $X = \mathbb{P}^2(\mathbb{C}) \supseteq \mathbb{P}^1(\mathbb{C}) = Y$

$$\begin{array}{ccc} H_2(\mathbb{P}^1(\mathbb{C}); \mathbb{Z}) & \longrightarrow & H_2(\mathbb{P}^2(\mathbb{C}); \mathbb{Z}) \\ \cong & & \\ \mathbb{Z} & & \\ 1 & \longmapsto & [\mathbb{P}^1(\mathbb{C})] \end{array}$$

generates $H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$

Psicological remark
When we have computed $H_q(\mathbb{P}^2(\mathbb{C}))$ and $H_q(\mathbb{P}^1(\mathbb{C}))$ we have seen
$H_2(Y) \simeq H_2(\mathbb{P}^2(\mathbb{C}))$
$\cong$
$H_1(\mathbb{C}^*) \simeq H_2(\mathbb{P}^1(\mathbb{C}))$

look back at the map

- Z algebraic curve of degree d in $\mathbb{P}^2(\mathbb{C})$, then

$[Z] = h\,[\mathbb{P}^1(\mathbb{C})]$ in $H_2(\mathbb{P}^2(\mathbb{C}); \mathbb{Z})$

which h? $\qquad h = d = \deg Z$

Idea: let $p \in \mathbb{P}^2(\mathbb{C})$ s.t.

$p \notin Z \qquad \pi_p|_Z : Z \longrightarrow \mathbb{P}^1(\mathbb{C})$

$$\deg(\pi_p|_Z) = \deg Z = d$$

$$\pi_{p_*} : H_2(Z) \longrightarrow H_2(\mathbb{P}^1(\mathbb{C}))$$

$$\pi_{p_*}([Z]) = \underbrace{\deg Z}_{d}\, [\mathbb{P}^1(\mathbb{C})]$$