Riemann surfaces and algebraic curves, Exercise sessions 1-13

1 Exercise session 1

(Exercise 1) Let $C = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = x^4 + y^4 + z^4 = 0\}$ be a plane curve in $\mathbb{P}^2_{\mathbb{C}}$.

- Show that C is a Riemann surface.
- Is the function $f = \frac{x}{y}$ holomorphic in $U_y \cap C$? Where $U_y = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : y \neq 0\}$.
- Which is a local coordinate for $U_y \cap C$ when $\frac{\partial F(x,1,z)}{\partial z} \neq 0$?
- Which are the intersection points $p_1, \ldots, p_n$ of $\{\frac{\partial F(x,1,z)}{\partial z} = 0\} \cap U_y \cap C$? Which is a local coordinate for this points? Write a local expression of a neighbourhood of a $p_i \in U_y \cap C \cap \{\frac{\partial F(x,1,z)}{\partial z} = 0\}$ as $\xi \mapsto \xi^k$. Explicitly write the value of k and how you choose the change of coordinates.
  Hint: it is asked to express $(x - p_i)$ as ξ^k. You may look at the proof of *Theorem* 1.4.3 in the notes to have a guide for the right procedure to follow.

(Exercise 2) Let $C = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = x^d + y^d + z^d = 0\}$ be a plane curve in $\mathbb{P}^2_{\mathbb{C}}$. Same questions as *Exercise* (1).

(Exercise 3) Let $C_0 = \{(x, y) \in \mathbb{C}^2 : y^2 - \prod_{i=1}^{2g+2}(x - a_i) = 0 : a_i \neq a_j \forall i \neq j\}$ be the intersection of a plane curve C in $\mathbb{P}^2_{\mathbb{C}}$ with $U_z = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : z \neq 0\}$.

- Show that C_0 is a Riemann surface.
- Homogenize the equation of C_0.
- Which are the points belonging to C for $z = 0$? Are those singular points?

(Exercise 4) Let $h(x)$ be a polynomial of degree $2g+2$ having distinct roots. Let us consider

$$U = \{(x, y) \in \mathbb{C}^2 : y^2 = h(x), x \neq 0\}.$$

Let us define $k(z) = z^{2g+2}h(1/z)$ and consider

$$V = \{(z, w) \in \mathbb{C}^2 : w^2 = k(z), z \neq 0\}.$$

Show that the map $\phi : U \to V$ sending (x, y) to $(1/x, y/x^{g+1}) = (z, w)$ is an isomorphism of Riemann surfaces.

2 Exercise session 2

(Exercise 1) Desingularise the curve of equation $w^2 + z^5 = 0$ in $\mathbb{C}^2$.

(Exercise 2) Let us consider the *tacnode* quartic D in $\mathbb{C}^2$ with equation $w(w - z^2) - z^4 = 0$. Show that D is desingularised after two blow-ups. (First blow-up in a neighbourhood U of $p = (0, 0)$ at p. Denote with $\pi : \tilde{U} \to U$ such blow-up. The strict transform C' of C via π has a nodal point at $q = ((0, 0), [1, 0]) \in \tilde{U}$. Then it is sufficient to blow-up $\tilde{U}$ at q.)

(Exercise 3) Desingularise the plane curve $C = \{[x : y : z] : z^{2g}y^2 - \prod_{i=1}^{2g+2}(x - a_i z) = 0 : a_i \neq a_j \forall i \neq j\}$ (see Exercise (3) of Exercise session 1).

(Exercise 4) Let us consider the same objects of Exercise (1) in Exercise sheet 1. Study $f = \frac{x}{y}$ in $C \cap \{y = 0\}$. Choose a local chart. Describe f in such chart. Does f have poles?

(Exercise 5) Let us consider the same objects of Exercise (2) in Exercise sheet 1. Study $f = \frac{x}{y}$ in $C \cap \{y = 0\}$. Choose a local chart. Describe f in such chart. Does f have poles?

3 Exercise session 3

(Exercise 1) Let us consider the curve

$$\Gamma = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} | xyz^3 + x^5 + y^5 = 0\}.$$

Let $\sigma : C \to \Gamma$ be the desingularisation of Γ. Let $p = [0 : 0 : 1]$.

- Show that $\sigma^{-1}(p)$ consists of two points $\sigma^{-1}(p) = \{p_1, p_2\}$.
- Let $f = x \circ \sigma$ and $h = y \circ \sigma$. Describe zeros, poles, ramification order, ect... of f, h and f/h.

(Exercise 2) Show that every complex torus $\mathbb{C}/L$ is isomorphic to a torus which has the form $\mathbb{C}(\mathbb{Z} + \tau\mathbb{Z})$, where τ is a complex number with strictly positive imaginary part.

4 Exercise session 4

Let Λ and Λ' be lattices $\{m\tau_1 + n\tau_2 : m, n \in \mathbb{Z}\}$ and $\{m\tau_1' + n\tau_2' : m, n \in \mathbb{Z}\}$.

(Exercise 1) Let $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ sending $z + \Lambda$ to $mz + b + \Lambda'$. Show that ϕ is a group homorphism iff $b \in \Lambda'$ iff $\phi(0) = 0$.

(Exercise 2) Let $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ sending $z + \Lambda$ to $-z + \Lambda$.

- Show that ϕ preserves the Weierstrass embedding of the complex torus in an elliptic curve E in $\mathbb{P}^2_{\mathbb{C}}$.
- Show that ϕ corresponds $\mathbb{P}^2_{\mathbb{C}}$ to $\psi : E \to E$ sending $[x : y : z]$ to $[x : -y : z]$.
- Show that the 2-torsion elements in $\mathbb{C}/\Lambda$ correspond to the points $[x : 0 : z]$ in E.

5 Exercise session 5

Let $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ be a lattice in $\mathbb{C}$ and let $\wp$ be the Weierstrass $\wp$-function for the lattice Λ.

(Exercise 1) Show that $\wp'$ and $\wp''$ do not have common zeros.

(Exercise 2) Describe in a neighbourhood of 0 the functions $\wp, \wp', \wp'', \wp^2, \frac{\wp'}{\wp^2}, \frac{\wp''}{\wp^2}$ in the form $F(z) = a_k z^k(1 + a_{k+1}z + ...)$ with $a_k \neq 0$ describing explicitly only a_k.

6 Exercise session 6

From **Exercise sheet 6**:
Let $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ be a lattice in $\mathbb{C}$ and let $\wp$ be the Weierstrass $\wp$-function for the lattice Λ. Let $g_2, g_3 \in \mathbb{C}$ such that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. The subset

$$C = \{[x : y : h : t] \in \mathbb{P}^3_{\mathbb{C}} : y^2 = 4xt - g_2xh - g_3h^2, x^2 = ht\},$$

is a submanifold of $\mathbb{P}^3_{\mathbb{C}}$.
The map

$$\mathbb{C}/\Lambda := T \to \mathbb{P}^3_{\mathbb{C}},$$

sending $z \mapsto [\wp(z) : \wp'(z) : 1 : \wp^2(z)]$ for $z \neq 0$ and $0 \mapsto [0 : 0 : 0 : 1]$ is a holomorphic map.

(Exercise 1) Prove that $\phi(T) = C$.

(Exercise 2) The map ϕ is injective and has maximal rank 1 on T.

7 Exercise session 7

Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and $\phi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ sending

$$(x : y : z) \mapsto (u : v : w) = (\lambda^2 x : \lambda^3 y : z),$$

(Exercise 1) Show that ϕ is biholomorphic.

(Exercise 2) Show that the elliptic curves C_1, C_2 with equations

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3, \quad v^2 w = 4u^3 - \lambda^4 g_2 u w^2 - \lambda^6 g_3 w^3$$

respectively, are isomorphic.

8 Exercise session 8

Let $X = \mathbb{C}/\Lambda$ be a complex torus.

(Exercise 1) Let D_1, D_2 be two divisors on X. Then $D_1 \sim D_2$ iff $deg(D_1) = deg(D_2)$ and $A(D_1) = A(D_2)$, where A is the Abel-Jacobi map (Definition 3.3.5 in the course pdf).

(Exercise 2) Let D be a divisor on X with $deg(D) > 0$. Then $dim(\mathcal{L}(D)) = deg(D)$.

Hint for Exercises:

(1) See Corollary 3.3.13 in the course pdf.

(2) As a Corollary of Abel's Theorem (Theorem 3.3.12 in the course pdf) we have the following:

Corollary 8.1 (Chapter V, Corollary 2.10 of Miranda'sbook). *Let D be a divisor on X with $deg(D) > 0$. Then D is linearly equivalent to a positive divisor. If $deg(D) = 1$, then $D \sim q$ for a unique point $q \in X$. If $deg(D) > 1$, then for any $x \in X$ there exists a positive divisor D' on X which is linearly equivalent to D and that does not contain x in its support.*

9 Exercise session 9

Let $C := \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = 0\}$, where $F(x, y, z)$ is a non singular polynomial of degree d, i.e. $(F, \partial F/\partial x, \partial F/\partial y, \partial F/\partial z) \neq (0, 0, 0, 0)$. Let $\{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : L(x, y, z) = 0\}$ be a line intersecting C in d distinct points. Without loss of generality, one can assume that $L(x, y, z) = x$.

(1) In the chart $C \cap \{z \neq 0\}$ show that x is a coordinate.

(2) Show that $w = \frac{x^{d-3}}{\frac{\partial F(x,y,1)}{\partial y}} dx$ is a holomorphic 1-form over $C \cap \{z \neq 0\}$.

(3) Prove that the genus of C equals $\frac{(d-1)(d-2)}{2}$ using what is known about $deg(div_0(\omega))$. (Hint: Proposition 3.2.7 and Corollary 3.2.8 of the course notes)

10 Exercise session 10

(Exercise 1) Let D be a divisor of a compact Riemann surface. Let $|D|$ be the complete linear system of D. Assume that $F = min\{E : E \in |D|\}$ in non-empty (F is the largest divisor that occurs in every divisor of D). It is clear that every divisor in $|D|$ can be written as the sum of F plus a divisor in $|D - F|$. Prove that $\mathcal{L}(D - F) = \mathcal{L}(D)$.

(Exercise 2) You may work on previous exercise sessions you have doubts about other you can start working on the next exercise sheet.

11 Exercise session 11

The following are exercises F and J of Problems $VI.2$ in [**Mira95**].

(Exercise 1) Let X be the Riemann sphere and let p be the point $z = 0$. Considering p as an ordinary divisor on X, show that $H^1(-p) = 0$ by explicitly finding preimages under α_{-p} for any Laurent tail divisor Z in $\mathcal{T}[-p](X)$.

(Exercise 2) Show that if $D_1 \sim D_2$, then $H^1(D_1) \simeq H^1(D_2)$, by showing that an isomorphism is induced from an appropriate multiplication operator on the corresponding Laurent tail spaces.

12 Exercise session 12

Let D denote a divisor on an algebraic curve X of genus g.

(Exercise 1) [**Mira95**][Problems VI.3 F.] Show that if Riemann-Roch is true for a divisor D then it is true for the divisor $K - D$.

(Exercise 2) (Stronger version of [**Mira95**][Problems VI.3 I.]) Let C be a non-singular curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$. The genus of C is:

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Show that the degree $(d-3)$ curves in $\mathbb{P}^2_{\mathbb{C}}$ cut out of C the canonical linear series, i.e.

$$|K| = \{\Gamma.C | \Gamma \text{ is a degree (d-3) curve in } \mathbb{P}^2_{\mathbb{C}}\},$$

where $\Gamma.C$ is defined in (2) of Exercise sheet 9.
Walkthrough the proof:

(Step 1) Fix a divisor $D = \Gamma.C$ where Γ is a degree $(d-3)$ curve in $\mathbb{P}^2_{\mathbb{C}}$ of equation $G(x, y, z) = 0$. Denote with W_{d-3} the space

$$\{(\frac{H}{G})\}_{|C} | H \text{ is a homogeneous polynomial of degree } (d-3)\}.$$

Prove that $W_{d-3} \subseteq \mathcal{L}(D)$.

(Step 2) Compute the dimension of W_{d-3}.

(Step 3) Compute the degree of D in terms of $g(C)$.

(Step 4) Compute the dimension of $\mathcal{L}(D)$.

(Step 5) Prove that $\mathcal{L}(D) = W_{d-3}$.

Hint for Step 3: Bèzout theorem (even if you have not seen a full proof yet, you can use it).
Hint for Step 4: modify the hypothesis of Exercise (3) in Exercise sheet 12, assuming that $dim\mathcal{L}(D) \geq g$; then the statement still holds! one may use this result to prove Step 4.

13 Exercise session 13

For a smooth plane cubic in $\mathbb{P}^2_{\mathbb{C}}$, define the j-invariant by taking it to a Weierstrass form.

(Exercise 1) Show that this is well defined.

(Exercise 2) Show that two cubics are isomorphic (one is taken to the other via a linear change of coordinate of $\mathbb{P}^2_{\mathbb{C}}$) iff the cubics have same j-invariant.

(Exercise 3) Show that for two isomorphic hyperelliptic cubic curves $E_1 \simeq E_2$, the branch points are isomorphic.

Fun application of Bèzout Theorem (Bonus Exercise), it would be explained better during the last exercise session (out of the purposes of the course):

1. Let C be a smooth plane curve of degree 4 in $\mathbb{P}^2_{\mathbb{R}}$. Prove that the topology of the pair $(C, \mathbb{P}^2_{\mathbb{R}})$ cannot realise, up to homeomorphism of $\mathbb{P}^2_{\mathbb{R}}$, the "shapes" depicted in the center and on the right of Fig. 1.

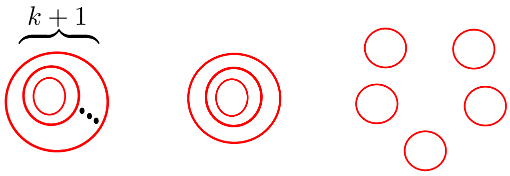

Figure 1:

2. Let C be a smooth plane curve of degree $2k$ in $\mathbb{P}^2_{\mathbb{R}}$. Prove that the topology of the pair $(C, \mathbb{P}^2_{\mathbb{R}})$ cannot realise, up to homeomorphism of $\mathbb{P}^2_{\mathbb{R}}$, the "shapes" depicted on the left of Fig. 1.