Riemann surfaces and algebraic curves, Exercise session 1

(Exercise 1) Let $C = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = x^4 + y^4 + z^4 = 0\}$ be a plane curve in $\mathbb{P}^2_{\mathbb{C}}$.

- Show that C is a Riemann surface.
- Is the function $f = \frac{x}{y}$ holomorphic in $U_y \cap C$? Where $U_y = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : y \neq 0\}$.
- Which is a local coordinate for $U_y \cap C$ when $\frac{\partial F(x,1,z)}{\partial z} \neq 0$?
- Which are the intersection points $p_1, \dots, p_n$ of $\{\frac{\partial F(x,1,z)}{\partial z} = 0\} \cap U_y \cap C$? Which is a local coordinate for this points? Write a local expression of a neighbourhood of a $p_i \in U_y \cap C \cap \{\frac{\partial F(x,1,z)}{\partial z} = 0\}$ as $\xi \mapsto \xi^k$. Explicitly write the value of k and how you choose the change of coordinates.
  Hint: it is asked to express $(x - p_i)$ as ξ^k. You may look at the proof of *Theorem* 1.4.3 in the notes to have a guide for the right procedure to follow.

(Exercise 2) Let $C = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : F(x, y, z) = x^d + y^d + z^d = 0\}$ be a plane curve in $\mathbb{P}^2_{\mathbb{C}}$. Same questions as *Exercise* (1).

(Exercise 3) Let $C_0 = \{(x, y) \in \mathbb{C}^2 : y^2 - \prod_{i=1}^{2g+2}(x - a_i) = 0 : a_i \neq a_j \forall i \neq j\}$ be the intersection of a plane curve C in $\mathbb{P}^2_{\mathbb{C}}$ with $U_z = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : z \neq 0\}$.

- Show that C_0 is a Riemann surface.
- Homogenize the equation of C_0.
- Which are the points belonging to C for $z = 0$? Are those singular points?

(Exercise 4) Let $h(x)$ be a polynomial of degree $2g + 2$ having distinct roots. Let us consider

$$U = \{(x, y) \in \mathbb{C}^2 : y^2 = h(x), x \neq 0\}.$$

Let us define $k(z) = z^{2g+2}h(1/z)$ and consider

$$V = \{(z, w) \in \mathbb{C}^2 : w^2 = k(z), z \neq 0\}.$$

Show that the map $\phi : U \to V$ sending (x, y) to $(1/x, y/x^{g+1}) = (z, w)$ is an isomorphism of Riemann surfaces.