UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

Loujean Cobigo
Matilde Manzaroli
Hannah Markwig

Exercises Geometric Group Theory

Summer semester 2022

Exercise sheet 1

Hand in exercises 1,2, 3a),3b),3c),4a) by April 27, 12 a.m. The remaining exercises will be carried out during the exercise session.
The exercise sheets can be delivered to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent to matilde.manzaroli 'at' uni-tuebingen.de.

A subgroup N of a group G is said *normal*, if $\forall g \in G$ and $\forall n \in N, gng^{-1} \in N$.

Exercise 1

(a) Show that N is a normal subgroup of G if and only if $gN = Ng$ for all $g \in G$, where $gN = \{gn \mid n \in N\}$ and $Ng = \{ng \mid n \in N\}$.

(b) Show that the image of a normal subgroup via a group homomorphism is not in general a normal subgroup, but the preimage of a normal subgroup via a group homomorphism is always a normal subgroup.

(c) Let $f : (G, \cdot) \to (H, \star)$ be a group homomorphism. Show that $\mathrm{Ker}(f)$ is normal and that the map

$$\begin{aligned} \tilde{f} : G/\mathrm{Ker}(f) &\to \mathrm{Im}(f) \\ [g] &\mapsto f(g) \end{aligned}$$

induced by f is a group isomorphism.

Exercise 2

Verify that the following function is a group homomorphism:

$$\begin{aligned} \exp : (\mathbb{R}, +) &\to (\mathbb{C}\backslash\{0\}, \cdot) \\ t &\mapsto \exp(2\pi i t). \end{aligned}$$

(a) Determine Ker(exp).

(b) To which subgroup of $(\mathbb{C}\backslash\{0\}, \cdot)$ is $\mathbb{R}/\mathrm{Ker}(\exp)$ isomorphic to and why?

Exercise 3

Let us consider the subset $H := \{\mathrm{id}, (12)\circ(34), (13)\circ(24), (14)\circ(23)\}$ of $\mathbb{S}_4$.

(a) Show that H is a subgroup of $\mathbb{S}_4$.

(b) Show that H is normal.

(*Hint: You may use without proving it that, for all $n \in \mathbb{N}_{>0}$ and Zykel $(x_1x_2 \dots x_k) \in \mathbb{S}_n$, one has that: $\sigma\circ(x_1 \dots x_k)\circ\sigma^{-1} = (\sigma(x_1) \dots \sigma(x_k))\ \forall \sigma \in \mathbb{S}_n$.*)

One can define the group $\mathbb{S}_4$ as the symmetry group of a equilateral tetrahedron with corners $1, 2, 3$ and 4. Let M_{AB} denote the middle-point of the segment $\overline{AB}$ for $1 \leq A, B \leq 4$, $A \neq B$. We define the straight lines $d_1 := M_{12}M_{34}$, $d_2 := M_{13}M_{24}$ and $d_3 := M_{14}M_{23}$ (see the picture).

(c) Explain geometrically: $(12)\circ(34)$ is a rotation around d_1, $(13)\circ(24)$ is a rotation d_2 and $(14)\circ(23)$ is a rotation around d_3.

To any permutation σ of $\mathbb{S}_4$ corresponds a bijection $\varphi_\sigma : \{d_1, d_2, d_3\} \to \{d_1, d_2, d_3\}$ determined by $\varphi_\sigma(M_{ij}M_{kl}) = M_{\sigma(i)\sigma(j)}M_{\sigma(k)\sigma(l)}$ for pairwise distinct $i, j, k, l \in \{1, 2, 3, 4\}$.

For example

$$\begin{aligned}
\varphi_{(12)}(d_1) &= \varphi_{(12)}(M_{12}M_{34}) = M_{21}M_{34} = d_1,\\
\varphi_{(12)}(d_2) &= \varphi_{(12)}(M_{13}M_{24}) = M_{23}M_{14} = d_3,\\
\varphi_{(12)}(d_3) &= \varphi_{(12)}(M_{14}M_{23}) = M_{24}M_{13} = d_2.
\end{aligned}$$

For each φ_σ one obtains in this way a permutation $\Phi(\sigma) \in \mathbb{S}_3$ uniquely defined (for example $\Phi((12)) = (23) \in \mathbb{S}_3$, because $\varphi_{(12)}$ exchanges d_2 with d_3 and fixes d_1). Let us define a map $\Phi : \mathbb{S}_4 \to \mathbb{S}_3$ sending $\sigma \mapsto \Phi(\sigma)$, where $\Phi(\sigma)$ has been constructed from some σ as previously shown.

(*Hint: The map Φ does the following: every $\sigma \in \mathbb{S}_4$ can be interpreted as a map from the tetrahedron onto itself (for example* (12) *is a reflexion with respect to the plane perpendicular to $\overline{12}$ through M_{12}. Then, such a map always sends any line d_i to some other d_j, with $i, j \in \{1, 2, 3\}$. Therefore this process produces the permutation $\Phi(\sigma)$.*)

(d) Show that for $\sigma = (34)$ and $\rho = (132)$ one has: $\Phi(\sigma \circ \rho) = \Phi(\sigma) \circ \Phi(\rho)$, i.e the homomorphism property of Φ applies to these particular σ and ρ.

The property shown in d) also holds for all choices of σ and ρ. You may use without proving it that Φ is a homomorphism.

(e) Show that Φ is surjective.

(f) Show that $\text{Kern}(\Phi) = H$ and conclude that this gives an isomorphism from $\mathbb{S}_4/H$ to $\mathbb{S}_3$.

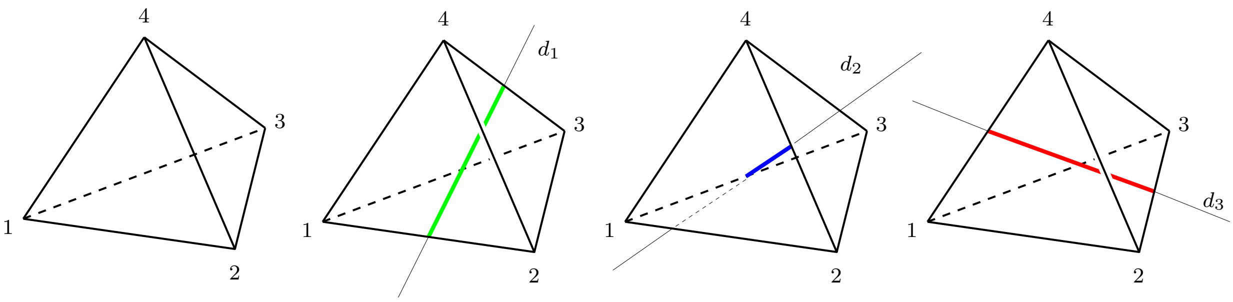

Abbildung 1: An equilateral tetrahedron with vertices $1, 2, 3, 4$ and the lines d_1, d_2, d_3.

Exercise 4

(a) Let G be the free group generated by $\{x, y\}$. Prove that the subgroup N of G generated by x^2, xyx^{-1}, y is a normal subgroup of index 2 in G. Of index 2 means that $|G/N| = 2$.

(b) Show that N is a free group generated by 3 elements.

The first exercise session will take place online on Wednesday, May 4, at 4:15-5:45 p.m. The link will be sent via email. From May 11th, the exercise class will take place in room C4H33 of the building C from 4:15 to 5:45 p.m.