Loujean Cobigo
Matilde Manzaroli
Hannah Markwig

UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

Exercises Geometric Group Theory
Summersemester 2022

Exercise sheet 10

Hand in exercises 1 (a), 2 by July 6, 11:59 a.m. The remaining exercises will be carried out during the exercise session.
Group 2 (respectively Group 1) can deliver the exercise sheets to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent via email to Manzaroli Matilde (respectively to Loujean Cobigo).

Exercise 1

(a) Show that the map $\varphi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, which is induced in each entry by the map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, is well-defined group homomorphism. Moreover show that $\mathrm{Ker}(\varphi) = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$.

(b) Conclude that the subgroup generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ has index 12 in $\mathrm{SL}_2(\mathbb{Z})$.

(Hint: For (a) the following division with a negative remainder could be helpful: For $x, y \in \mathbb{Z}$ and $y \neq 0$ we can write $x = yq + r$ for suitable r, q with $|r| \leq \frac{1}{2}|y|$.)

Exercise 2
Let $\mathbb{R}^2 \backslash \{0\}$ be equipped with the standard metric of $\mathbb{R}^2$. Show that the following group operation of $\mathbb{Z}$ on $\mathbb{R}^2 \backslash \{0\}$

$$\mathbb{Z} \times \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2 \backslash \{0\}, \left(n, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto n \cdot \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 2^n x \\ 2^{-n} y \end{pmatrix}$$

is not proper.

Exercise 3
Let $a \in \mathbb{R}_{>1}, a' \in \mathbb{R}_{\geq 0}$ and $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto a^x$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto x^{a'}$ be generalised growth functions. Show that $g \prec h$ and $h \not\prec g$ hold.