UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

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Exercises Geometric Group Theory

Summersemester 2022

Exercise sheet 12

Hand in exercises 1, 3 a) by July 20, 11:59 a.m. The remaining exercises will be carried out during the exercise session.
Group 2 (respectively Group 1) can deliver the exercise sheets to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent via email to Manzaroli Matilde (respectively to Loujean Cobigo).

Exercise 1

Show that the free group $\mathbb{F}_n$ of rank $n \geq 2$ is virtually solvable.

Exercise 2

Let G be a finitely generated group of polynomial growth and let $\pi : G \to \mathbb{Z}$ be a surjective group homomorphism. Show that $\mathrm{Ker}(\pi)$ is finitely generated.

(*Hint: Choose $g \in G$ such that $\pi(g) = 1$ and show that fur such g there exists a finite set $S \subset \mathrm{Ker}(\pi)$, whose union with $\{g\}$ already generates G. In order to prove the statement it suffices to find $N \in \mathbb{N}$ such that the set $\{g^n s g^{-n} | s \in S, n \in \{-N, \ldots, N\}\}$ generates the kernel of π.*)

Let X be a connected graph. If one replaces all edges of X with straight lines of length 1 and defines the graph metric accordingly, then the resulting metric space is called the *geometric realization of X*. For this geometric realization one writes $\left(|X|, d_{|X|}\right)$. For example, the geometric realization of $\mathrm{Cay}\left(\mathbb{Z}, \{1\}\right)$ is isometric to $\mathbb{R}$ with the standard metric.

Exercise 3

Let $X = (V, E)$ be a connected graph.

(a) Show that the geometric realization of X is geodesic.

(b) Show that the canonical embedding $\varphi : V \to |X|$, where V is equipped with the minimal path length metric (Defnition 5.2.1 in [1]), is an isometric embedding and a quasi-isometry.

Literatur

[1] Geometric Group Theory: An Introduction, Clara Löh.