UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

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Exercises Geometric Group Theory

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Exercise sheet 2

Hand in exercises 1,2 by May 4, 11:59 a.m. The remaining exercises will be carried out during the exercise session.
The exercise sheets can be delivered to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent to Matilde Manzaroli or Loujean Cobigo via mail.

Exercise 1

Let us define a subgroup $\mathbb{A}_4$ of $\mathbb{S}_4$ as follows

$$\mathbb{A}_4 := \{\mathrm{id}, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}.$$

Let H be the subgroup of $\mathbb{S}_4$ defined in Exercise sheet 1, exercise 3.

(a) Show that H is a normal subgroup of $\mathbb{A}_4$.

(b) Show that any 2-element subgroup of H is a normal subgroup of H.

(c) Conclude that being a normal subgroup is not a transitive property.

Exercise 2

Show that $\mathbb{S}_3$ can be represented via the following generators and relations

$$\mathbb{S}_3 \cong \{\tau, \sigma | \sigma^3 = \tau^2 = \sigma\tau\sigma\tau = \mathrm{id}\}. \quad (1)$$

(*Hint: Write the right side of (1) as the quotient of a free group by the kernel of an opportune morphism and use exercise 1 in Exercise sheet 1.*)

Exercise 3

(a) Let N be a normal subgroup of G and let U be a subgroup of G, such that $N \cap U = \{\mathrm{id}\}$ and $NU = G$. Show that $G \cong N \rtimes_\phi U$, where $\phi : U \to \mathrm{Aut}(N)$ sends u to k_u which is the conjugation with u.

(b) Let U be a subgroup of $\mathbb{S}_4$, which consists of exactly the permutations that fix 4 and let H be the subgroup of $\mathbb{S}_4$ defined in exercise 3 of Exercise sheet 1. Show that $\mathbb{S}_4 \cong H \rtimes_\varphi U$ for an opportune φ.

(c) Find another subgroup $U' \neq U$ von $\mathbb{S}_4$, such that $\mathbb{S}_4 \cong H \rtimes_{\varphi'} U'$ for an opportune φ' and conclude that the decomposition from (a) is not unique for a given normal subgroup N.

(*Hint: First show for (a) that for $g \in G$ there exists a unique $n \in N$ and a unique $u \in U$ such that $g = nu$. For (b) you should take a look at problem 3, sheet 1 and consider to which group U might be isomorphic. In (c) you can save yourself some work by choosing U' in such a way that you can take arguments from (b).*)
