UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

Loujean Cobigo
Matilde Manzaroli
Hannah Markwig

Exercises Geometric Group Theory

Summersemester 2022

Exercise sheet 3

Hand in exercises 1,2 by May 11, 11:59 a.m. The remaining exercises will be carried out during the exercise session.
Group 2 (respectively Group 1) can deliver the exercise sheets to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent via email to Manzaroli Matilde (respectively to Loujean Cobigo).

If a group (or a subgroup) is finite, then $|G|$ denotes the *order* of G.

The *order* of an element g of a group G is defined as the minimum of all $m \in \mathbb{N}_{>0}$, such that $g^m = \mathrm{id} \in (G, \cdot)$.

Exercise 1

(a) Let g be an element of a finite group G. Show that the order of g divides the order of G.

(b) If G is a finite group of even order, then there exists $g \in G$ such that $\mathrm{Ord}(g) = 2$.

(c) Show that the alternating group $\mathbb{A}_4$ (defined in exercise 1 in Exercise sheet 2) has no subgroup of order 6 and deduce that the converse of exercise 1 in Exercise sheet 3 does not hold.

(*Hint: For (a) choose a suitable* $x \in \mathbb{N}$*, such that* $\{\mathrm{id}, g, \ldots, g^x\}$ *forms a subgroup of* G*.*)

Exercise 2

Assume that there exists a free amalgamated product and show that this product is unique up to canonical isomorphism.

(*Hint: Retrace the steps of the proof of uniqueness of a free group.*)

Let A, G_1, G_2 be groups and α_1, α_2 injective group homomorphism, where $\alpha_i : A \to G_i$. Let S_i be a set of the right cosets representative ${}^{G_i}/_{A}$, where we identify A via α_i with its respective image in G_i such that $\mathrm{id}_{G_i} \in S_i$. For $g \in G_1 \star_A G_2$ and a finite sequence $i = (i_1, \ldots, i_n)$ in $\{1, 2\}$ with $n \geq 0$ and $i_m \neq i_{m+1}$ with $1 \leq m \leq n-1$ we call g a *reduced word* of type i, if there exists $a \in A, s_1 \in S_{i_1} \backslash \{\mathrm{id}\}, \ldots, s_n \in S_{i_n} \backslash \{\mathrm{id}\}$ such that $g = a s_1 \cdots s_n$. Then the set of all reduced words X is defined as the disjoint union of X_i, where X_i denotes the set of all reduced words in $G_1 \star_A G_2$ of type i. One can show that X is a group with merging and reducing words and that $X = G_1 \star_A G_2$.

Exercise 3

(a) Reduce the word $g = (1432)(123)(acbd)(1423)(ad)(bc) \in \mathbb{S}_4 \star_{\mathbb{S}_3} \mathbb{S}_4$, where we distinguish the two copies of $\mathbb{S}_4$ by considering the left $\mathbb{S}_4$ as bijections to $\{1, \ldots, 4\}$ and the right $\mathbb{S}_4$ as bijections on $\{a, \ldots, d\}$. The α_i's are given by the canonical embedding (fix 4 respectively d)) of $\mathbb{S}_3$ in $\mathbb{S}_4$. Use $\mathrm{id}, (14), (24)$ and (34) as right coset representatives.

(b) Show that every word in $G_1 \star_A G_2$ can be uniquely reduced.

Exercise 4

Show that $\mathbb{Z}_4 \star_{\mathbb{Z}_2} \mathbb{Z}_6 \cong \mathrm{SL}_2(\mathbb{Z})$.

(*Hint: Identify* $\mathbb{Z}_4$ *with the product of* $A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $\mathbb{Z}_6$ *with the product of* $B := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$*. Also, it might be helpful to look at the matrix* $C := B^2$*.*)
