UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

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Exercises Geometric Group Theory

Summersemester 2022

Exercise sheet 4

Hand in exercises 2 and 3 (only for graphs A,B,D) by May 18, 11:59 a.m. The remaining exercises will be carried out during the exercise session.
Group 2 (respectively Group 1) can deliver the exercise sheets to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent via email to Manzaroli Matilde (respectively to Loujean Cobigo).

The *genus* of a finite connected graph G is defined as $g := 1 - \#V + \#E$, where V is the set of vertices of G and E the set of edges of G.

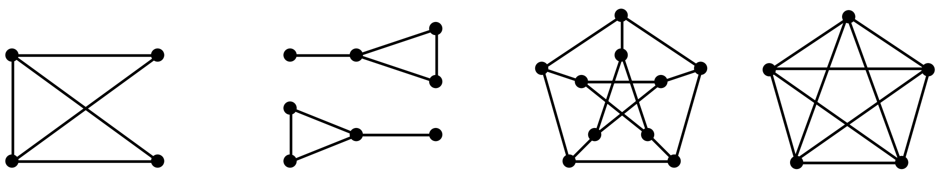
Exercise 1

Show that the genus of a finite connected graph is always non negative and that is zero if and only if the graph is a tree.

Exercise 2

(a) Draw the Cayley graphs of $\mathbb{Z}_8$ with respect to the generating sets $\{1\}$ and $\{2, 3\}$.

(b) Let S be a generating set of $\mathbb{Z}_n$ and $k \in S$, such that $\frac{n}{gcd(k,n)} > 2$. Show that the Cayley graph of $\mathbb{Z}_n$ with respect to S has exactly $gcdT(k, n)$ disjoint $\frac{n}{gcd(k,n)}$-cycles.



A B C D

Exercise 3

(a) Determine the automorphism groups of the graphs shown above.

(b) Determine all subgraphs of A that are Cayley graphs and respectively give a group that belongs to this Cayley graph.

Bestimmen Sie alle Untergraphen von A, die Cayley Graphen sind und geben Sie jeweils eine Gruppe an, die zu diesem Cayley Graphen gehört.

(c) Which of the above graphs are Cayley graphs?

(*Hint: In order to determine the automorphism group of C,it might be helpful to identify the vertices of C with 2-element subsets of $\{1, \dots, 5\}$. For (c) you may use that all groups of order 10 are either isomorphic to $\mathbb{Z}_{10}$ or D_5, where D_n is the nth dihedral group.*)

Exercise 4

- Show that D_4 is $\mathbb{Z}/4\mathbb{Z} \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$, where $\phi : \mathbb{Z}/2\mathbb{Z} \to Aut(\mathbb{Z}/4\mathbb{Z})$.
- Show that D_n is $\mathbb{Z}/n\mathbb{Z} \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$, where $\phi : \mathbb{Z}/2\mathbb{Z} \to Aut(\mathbb{Z}/n\mathbb{Z})$.
