UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

Loujean Cobigo
Matilde Manzaroli
Hannah Markwig

Exercises Geometric Group Theory

Summersemester 2022

Exercise sheet 5

Hand in exercises 1, 2, 3a) by May 25, 11:59 a.m. The remaining exercises will be carried out during the exercise session.
Group 2 (respectively Group 1) can deliver the exercise sheets to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent via email to Manzaroli Matilde (respectively to Loujean Cobigo).

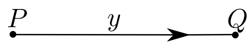
Exercise 1

Show that there is a fixed point when operating a group of order 55 on a set with 39 elements.

Exercise 2

Let G be the free group of two generators x, y and N be the subgroup defined on Sheet 1 in Exercise 4. Sketch the spanning tree of the operation by multiplying N on the Cayley graph of $(G, \{x, y\})$.

Let G be a group that operates on a *directed* graph X. Make sure that Cayley graphs can be viewed as directed graphs and that groups also operate on directed graphs. The *quotient graph* ${}^{G}/_{X}$ of X with respect to G is the graph that results from X by identifying vertices and edges of X in the same orbit. For example, spanning trees of group operations on graphs are quotient graphs. A subgraph $T \subset X$ is called *fundamental domain* of X with respect to G if there exists a graph isomorphism $T \to {}^{G}/_{X}$. Remark that $G.T = X$. A graph is called *segment* if it is isomorphic to the graph shown below with vertices P, Q and edge y.



Exercise 3

Let G be a group, that operates on a directed graph X in such a way that there exists a fundamental domain T of X with respect to G and such that T is a segment. Let G_P, G_Q, G_y be the stabilizers of the group operation of G on X. Let G_P, G_Q, G_y be the stabilizers of the group operation from G to X of the vertices $P, Q \in V(X)$, respectively the edge $y \in E(X)$, of the segment $T \subset X$. Show that:

(a) The graph X is connected if and only if G is generated by $G_P \cup G_Q$.

(b) The graph X contains no cycles if and only if the group homomorphism $G_P \star_{G_y} G_Q \to G$ induced by the inclusions $G_P \to G$ and $G_Q \to G$ is injective.

(c) The graph X is a tree if and only if the map $G_P \star_{G_y} G_Q \to G$ from (b) is a group isomorphism.

(Hint: For (b) you may use without proof that for $g_i \in G_{P_i} \backslash G_y$ with $0 \leq i \leq n$, $\{P_i, P_{i+1}\} = \{P, Q\}$ and $g_0 \cdots g_n = \mathrm{id}_G$ it follows that $g_0 \cdots g_n \neq \mathrm{id}_{G_P \star_{G_y} G_Q}$.)

Exercise 4

Let G_1, G_2 be groups and $G := G_1 \star G_2$ be the free group of such groups. Show that there exists a tree X on which G operates and with fundamental domain T such that T is a segment and $G_P = G_1, G_Q = G_2$ and $G_y = \{id\}$ (same notations as the previous exercise).
