

Exercises Geometric Group Theory
Sommersemester 2022

Exercise sheet 6

Hand in **exercises 1(a), 4 and 5** by June 1, 11:59 a.m. The remaining exercises will be carried out during the exercise session.

Group 2 (respectively **Group 1**) can deliver the exercise sheets to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent via email to Manzaroli Matilde (respectively to Loujean Cobigo).

Exercise 1

- (a) Show that for a free group \mathbb{F}_n of rang $n \geq 2$ and $m \in \mathbb{N}$, there exists a subgroup $U \subset \mathbb{F}_n$, which is a free group of rang $r \geq m$.
- (b) Show that $\forall m \in \mathbb{N}$, there exists a subgroup $U \subset \mathbb{F}_2$, which is free and of rang m .

(Hint: Exercise 1 (b): look at exercise sheet 1 exercise 4.)

Exercise 2

Show the following Ping-Pong Lemma of free products: Let G be a group and let $G_1, G_2 \subset G$ be subgroups such that G is generated by $G_1 \cup G_2$ with $|G_1| \geq 3, |G_2| \geq 2$. Let X be a set on which G operates. Let X_1, X_2 non-empty subsets of X with $X_2 \not\subset X_1$ and

- (1) $\forall g \in G_1 \setminus \{id\} : gX_2 \subset X_1$,
- (2) $\forall g \in G_2 \setminus \{id\} : gX_1 \subset X_2$.

Then $G \cong G_1 \star G_2$.

(Hint: Look at exercise sheet 3, recall reduced words in free products.)

Exercise 3

Let d_1, d_2 be the metrics defined as follows.

$$d_i \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right) := \sqrt[i]{|x - x'|^i + |y - y'|^i}$$

Show that (\mathbb{R}^2, d_1) is Bilipschitz equivalent to (\mathbb{R}^2, d_2) .

Exercise 4

Decide whether \mathbb{Z} and $\{n^3 | n \in \mathbb{Z}\}$ are quasi-isometric with respect to the Euclidean metric on \mathbb{Z} and justify your answer.

Exercise 5

Let X, Y, Z be metric spaces and $f : X \rightarrow Y, g : Y \rightarrow Z$ quasi-isometries. Show that $g \circ f$ is a quasi-isometry.
