UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

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Exercises Geometric Group Theory

Summersemester 2022

Exercise sheet 9

Hand in exercises 1, 3 by June 29, 11:59 a.m. The remaining exercises will be carried out during the exercise session.
Group 2 (respectively Group 1) can deliver the exercise sheets to Manzaroli mailbox in room A16 building C on the 3rd floor or scanned and sent via email to Manzaroli Matilde (respectively to Loujean Cobigo).

Exercise 1

Show that commensurability is an equivalence relation.

Let G be a topological group and $G_i \subset G$ (with $i \in I$ any index set) be disjoint non-empty subsets of G, which are open and closed and such that $G = \bigcup_{i \in I} G_i$. Moreover the G_i's are maximal, i.e. $\emptyset$ and G_i are the only sets with respect to the subspace topology of G_i that are open and closed at the same time. The G_i's are called *connected components* of G. This decomposition of a topological group into connected components always exists and is unique.

Exercise 2

Let G be a topological group. Let us considers its unique decomposition in connected components $\bigcup_{i \in I} G_i$. Show that if the connected component id $\in G_0$ then G_0 forms a subgroup of G.

Exercise 3

Let $G = H = \mathbb{Z}$ be groups which operate on $\mathbb{R}^2$ as follows :

$$G \times \mathbb{R}^2 \to \mathbb{R}^2, (n, (x, y)) \mapsto (n + x, y) \text{ und } \mathbb{R}^2 \times H \to \mathbb{R}^2, ((x, y), m) \mapsto (x, y + m).$$

Show that $\mathbb{R}^2$ with the above group operation is a set-theoretic coupling of G and H.

Exercise 4

Let G be a topological group and $\Gamma \subset G$ a closed discrete subgroup. Show that the left multiplication $\Gamma \times G \to G$ is a proper group operation.