Exercises

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Example 0.1. *Let C be a non-singular real algebraic plane curve of degree d. Show that:*

- *If d is odd, then $C(\mathbb{R})$ has exactly one pseudo-line;*
- *If d is even, then $C(\mathbb{R})$ has only ovals (also no ovals is possible).*

Example 0.2 (Harnack's inequality)**.** *Let C be a non-singular real algebraic plane curve of degree d. Denote with l the number of the connected components of $C(\mathbb{R})$. Prove that*

$$l \leq \frac{(d-1)(d-2)}{2} + 1$$

applying Bézout theorem.

Example 0.3. *Let us consider $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ equipped with the anti-holomorphic involution*

$$\sigma : ([x_0 : y_0], [x_1 : y_1]) \mapsto ([\overline{x_1} : \overline{y_1}], [\overline{x_0} : \overline{y_0}])$$

. Which anti-holomorphic involution on $\mathbb{P}^3(\mathbb{C})$ make the Segre embedding

$$i : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \hookrightarrow \mathbb{P}^3(\mathbb{C})$$

$$([x_0 : y_0], [x_1 : y_1]) \mapsto [x_0x_1 : x_0y_1 : y_0x_1 : y_0y_1]$$

a real map?
Recall: Given $f : \sigma_X \circlearrowleft X \to Y \circlearrowleft \sigma_Y$ between real algebraic varieties, we say that f is real if

$$f \circ \sigma_X = \sigma_Y \circ f$$

Example 0.4.

- *Let $C \circlearrowleft \sigma$ be a smooth projective real algebraic curve of genus g. Show that, if C is maximal, then it is separating.*
- *Let $C \circlearrowleft \sigma$ be a smooth projective real algebraic plane curve of genus g. Show that, if C has a maximal nest, then it is separating.*
  Hint: show there exists a map f

  $$\sigma \circlearrowleft C \xrightarrow{f} \mathbb{P}(\mathbb{C}) \circlearrowleft conj$$

  such that

  1. *f is real*
  2. *f has degree d*
  3. *$f^{(-1)}\mathbb{P}(\mathbb{R}) = C(\mathbb{R})$.*

Example 0.5.

1. *Via Harnack's construction method construct inductively non-singular real maximal plane curves of degree d.*

   Hint for the induction step: assume that there exists a non-singular real algebraic plane curve A_{n-1} of degree $n-1$ such that:

   - *exactly one connected component of $A_{n-1}(\mathbb{R})$ intersects a given real line in $n-1$ real points;*
   - *A_{n-1} is maximal.*

2. *Which real schemes may a real maximal plane curve of degree d realise?*

Example 0.6. *Classify non-singular real algebraic plane curves of degree for all $d \leq 5$.*