# CORRECTION EX SHEET 8 :

1) $\Lambda$ generated by $1$ and $\tau$

$$\xi = a\tau + b \qquad a, b \in \mathbb{R}$$

$$\lambda = p\tau + q \qquad p, q \in \mathbb{Z}$$

CLAIM :

• $$\boxed{\vartheta_\xi(z+\lambda) = e_\xi(\lambda, z)\,\vartheta_\xi(z)}$$

• $$\boxed{e_\xi(\lambda, z) = e^{\pi i(-p^2\tau + 2a\lambda - 2p(z+\xi))}}$$

RK : $\vartheta$ INVARIANT UNDER

- $\mathbb{Z}$-shifts
- $\mathbb{Z}\tau$-shifts

**Note** ( Proof by induction of $\mathbb{Z}\tau$-invariance of $\vartheta$ )

(1) $m\tau$ $\quad m \geq 0$ ✓ $\quad m \in \mathbb{N}$

(2) $m\tau$ $\quad m < 0$ : use (1)

$$\vartheta(z) = \vartheta((z+m\tau) - m\tau)$$

$$= e^{-\pi i (m^2\tau - 2m(z+m\tau))} \vartheta(z+m\tau)$$

**Recall** ( around def. 3.3.9 in the pdf course )

$$\vartheta(z+a\tau+b) = e^{-\pi i (a^2\tau + 2a(z+b))} \cdot \vartheta_\xi(z)$$

$$\vartheta_\xi(z) := \sum_{m\in\mathbb{Z}} e^{\pi i ((m+a)^2\tau + 2(m+a)(z+b))}$$

$$\vartheta_\xi(z+\lambda) \overset{\text{DEF}}{=} e^{\pi i(a^2\tau+2a(z+\lambda+b))}\,\vartheta(z+\lambda+\xi)$$

$$\boxed{\begin{aligned}\xi &= a\tau+b\\ \lambda &= p\tau+q\end{aligned}}$$

Invariance under $\mathbb{Z}$-shifts

$$= \underbrace{e^{\pi i(a^2\tau+2a(z+\lambda+b))}}_{A}\,\vartheta(z+\xi+p\tau)$$

Invariance under $\mathbb{Z}\tau$-shifts

$$= A\cdot \underbrace{e^{-\pi i(p^2\tau+2p(z+\xi))}}_{B}\,\vartheta(z+\xi)$$

DEF. $\vartheta_\xi(z)$

$$= \underline{A\cdot B\, e^{-\pi i(a^2\tau+2a(z+b))}}\,\vartheta_\xi(z)$$

$$= \underbrace{e^{2\pi i a\lambda-\pi i p^2\tau-2\pi i p(z+\xi)}}_{e_\xi(\lambda, z)}\,\vartheta_\xi(z)$$

Psicological rK:

$$\mathcal{L}(D) = \{ f \in \mathcal{M}(X) : \operatorname{div}(f) \geq -D \}$$

$D$ divisor on $X$ Riemann surf.

- $D = D^+ - D^-$ $\quad D^+, D^-$ positive divisors
- $\operatorname{div}(f) = \operatorname{div}_0(f) - \operatorname{div}_\infty(f)$

○ $\operatorname{div}_0(f), \operatorname{div}_\infty(f)$ have disjoint support respect. $D^+, D^-$

$\Rightarrow$
- $\operatorname{div}_0(f)$ must be "better" than the zeros of $D^-$
- $\operatorname{div}_\infty(f)$ have poles "not worse" than the zeros of $D^+$

Correction ex 2 by Lena and Dennis

$$C = \{ F(xyz) = 0 \} \subseteq \mathbb{P}^2_{\mathbb{C}}$$ $\deg F = d$, $F$ non-sing

$\phi(x,y,z)$ $\deg \phi = d-3$ $\phi$ non-sing

$$\omega = \frac{\phi(xyz) z^2}{\frac{\partial F}{\partial y}(xyz)} d\left(\frac{x}{z}\right)$$

<u>CLAIM</u>: $\omega$ is holomorphic in $C \cap \{z \neq 0\}$

<u>Proof</u>: in $C \cap \{z \neq 0\}$

$$\omega = \frac{\phi(x,y,1)}{\frac{\partial F}{\partial y}(x,y,1)} dx$$

1) If $\frac{\partial F}{\partial y}(x,y) \neq 0 \Rightarrow x$ local coord and $\omega$ is holomorphic!

2) if $\frac{\partial F}{\partial y}(x,y) = 0$, we can apply the implicit function theorem

$y$ is coordinate $\forall (x,y)$ s.t. $\frac{\partial F}{\partial y}(x,y) = 0$ and $x = g(y)$

for some holomorphic $g$.

then $$\omega = \frac{\phi(g(y), y, 1)}{\frac{\partial F}{\partial y}(g(y), y, 1)} dg(y)$$

$$dg(y) = -\frac{1}{\frac{\partial F}{\partial x}(g(y), y, 1)} \cdot \frac{\partial F}{\partial y}(g(y), y, 1) dy$$

$$\Rightarrow \boxed{\omega = -\frac{\Phi(g(y), y, 1)}{\frac{\partial F}{\partial x}(g(y), y, 1)}\, dy}$$

$\Rightarrow$ $\omega$ is holomorphic

( $y$ coordinate and $\frac{\partial F}{\partial x} \neq 0$ )

Psicological rk:

Ex SESSION 8

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D divisor on X Riemann surf.

- $D = D^+ - D^-$ $\quad D^+, D^-$ positive divisors
- $\operatorname{div}(f) = \operatorname{div}_0(f) - \operatorname{div}_\infty(f)$

○ $\operatorname{div}_0(f), \operatorname{div}_\infty(f)$ have disjoint support respect. $D^+, D^-$

$\Rightarrow$ • $\operatorname{div}_0(f)$ must be "better" than the zeros of $D^-$

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D divisor on X

<u>RC</u>: $|D| := \{ E \in \mathrm{Div}(X) : E \geq 0 \text{ s.t. } E \sim D \}$

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<u>ex 1</u>: (use Abel's theorem)

Let $X = \mathbb{C}/\Lambda$ cx torus.
Let $D_1, D_2$ be two divisors on $X$. then $D_1 \sim D_2$ iff $\deg(D_1) = \deg(D_2)$ and $A(D_1) = A(D_2)$, where $A$ is the Abel–Jacobi map (Def. 3.3.5)

<u>ex 2</u>: Let $X = \mathbb{C}/\Lambda$ cx torus
Let $D$ be a divisor on $X$ with $\deg(D) > 0$. Then

$$\dim(\mathcal{L}(D)) = \deg(D)$$

HINT:

From Abel's theorem

COR (Miranda, chapter V, Cor. 2.10)

Let $D$ be divisor on $X$ with $\deg(D) \geq 0$. Then $D$ is lin. equivalent to a positive divisor.

If $\deg(D) = 1$, then $D \sim q$ for a unique point $q \in X$.

If $\deg(D) > 1$, then $\forall x \in X$ $\exists$ positive diviser $E$ on $X$ which is lin. eq. to $D$ and s.t. $E$ does not contain $x$ in its support.