This is an aside chapter to introduce you to SINGULAR HOMOLOGY

we will not need the all extent of it, but singular homology is useful in math life! therefore we present this tool in details. The idea is to assocate to a topological space some (abelian) groups encoding some features and properties of such topological space.

A possible reference (but it is plenty of books about this subject) is

Algebraic topology, Allen Hatcher

Singular homology

§ Abelian groups and exact sequences

Def: • $(G,+)$ abelian group

Finitely generated if

$\exists \{g_i\}_{i\in I}$ finite

s.t. $\forall g \in G \ \exists m_i \in \mathbb{Z}$

s.t. $g = \sum_{i\in I} m_i g_i$

• We say that $(G,+)$ is free if

$\sum_{i\in I} m_i g_i = 0$ iff $m_i = 0 \ \forall i \in I$

• $B = \{b_i\}_{i\in J}$ is a basis of $(G,+)$ if it is free and it is a set of generators.

Properties: $(G,+,B)$ free abelian group with a finite basis B

$\forall \varphi : B \longrightarrow G'$ abelian group

$$\begin{array}{cc} \downarrow & \exists!\ \tilde{\varphi} \text{ s.t.} \\ G & \tilde{\varphi}|_B = \varphi \end{array}$$

<u>Def</u> : $g \in G$ is a <u>torsion element</u> if $\exists\ m \in \mathbb{Z}_{\neq 0}$ s.t. $mg = 0$

$T(G)$ = all torsion element of G
<u>torsion subgroup of G</u>

<u>Properties</u> (exercise)

G, G' two abelian groups

1) $\varphi : G \to G'$ then $\varphi(T(G)) \subseteq T(G')$
2) φ isomorphism $\Longrightarrow T(G) \simeq T(G')$

<u>FACT</u> : G abelian group finitely generated then $T(G)$ is finite and

$$G \simeq \mathbb{Z}^m \oplus T(G)$$

Fix a commutative ring with unit A

Def: • $(G,+)$ abelian group

$$A \times G \xrightarrow{\cdot} G \quad \text{s.t.} \quad \forall\, g, g' \in G,\ a, b \in A$$

1. $a \cdot (g+g') = ag + ag'$
2. $(a+b) \cdot g = ag + bg$
3. $(ab) \cdot g = a \cdot (b \cdot g)$
4. $1 \cdot g = g$

We say G is an A-module.

• Let G be finitely generated. We say that G is a FREE A-MODULE if G has a basis (G is isomorphic to $\bigoplus_n A$)

• $\mathbb{Z}$-module $(\equiv)$ abelian group

We need a bit more generality because we are interested mostly with $\mathbb{Z}$-modules and $\mathbb{Z}/2\mathbb{Z}$-modules.

Def: • $\cdots \to G_{n-1} \xrightarrow{f_{n-1}} G_n \xrightarrow{f_n} G_{n+1} \to \cdots$ a sequence of a A-modules and module homomorphism if $\boxed{\operatorname{Im} f_{i-1} = \ker f_i \ \forall i}$

• A short exact sequence is an an exact sequence of the form

$$0 \to H \xrightarrow{f} G \xrightarrow{g} K \to 0$$

RK: f is injective ($\ker f = 0$)
g is surjective ($\operatorname{Im} g = K$)

Lemma: Let $0 \to D \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of A-modules.

Then TFFAE (the following facts are equivalent)

1. $\exists\ h : B \to D$ st. $h \circ f = id_D$
2. $\exists\ k : C \to B$ st. $g \circ k = id_C$
3. $\exists$ isomorphism $\varphi : B \to D \oplus C$ making the following diagram commute

$$\begin{tikzcd}
 & & B \arrow[dd, "\varphi"] \arrow[drr, "g"] & & & \\
0 \arrow[r] & D \arrow[ur, "f"] \arrow[dr, "i_D"'] & & & C \arrow[r] & 0 \\
 & & D \oplus C \arrow[urr, "p_C"'] & & &
\end{tikzcd}$$

where i_D is the inclusion

p_C is the projection

Such exact sequence is called SPLIT

Proof

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r, "f"] & B \arrow[r, "g"] \arrow[l, bend left, dashed, "h"] & C \arrow[r] \arrow[l, bend left, dashed, "k"] & 0
\end{tikzcd}$$

3 $\Rightarrow$ 1, 2

$h := p_D \circ \varphi$

$k := \varphi^{-1} \circ i_C$

1 $\Rightarrow$ 3

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r, "f"] & B \arrow[r, "g"] \arrow[l, bend left, "h"] & C \arrow[r] & 0
\end{tikzcd}$$

$\operatorname{Im} f = \ker g$

$h \circ f = id_D$

Define $\varphi : B \longrightarrow D \oplus C$ as
$b \longmapsto (h(b), g(b))$

φ injective:

(a) $\ker \varphi = \ker h \cap \ker b = \ker h \cap \operatorname{Im} f$

(b) Remark that $B = \operatorname{Im}(f) \oplus \ker(h)$

in fact $\forall b \in B \quad b = b - f(h(b)) + \underbrace{f(h(b))}_{\in \operatorname{Im} f}$

and $b - fh(b) \in \ker h$

$(h(b) - id_D h(b)) = 0$

(a) + (b) $\Longrightarrow$ φ injective

φ surjective: $(d, c) \in D \oplus C$

g is surjective $\Longrightarrow \exists\, b$ s.t. $g(b) = c$

Q: $h(b) = b$?

$$\boxed{b = f(d') + b'}$$
||

$$c = g(\cancel{f(a')}) + g(b')$$

$\mathrm{Im} f = \ker g$

$$\boxed{h\big(f(a) + b'\big) = a} \implies$$

$$\tilde{b} = f(a) + b'$$

$$\varphi(\tilde{b}) = (a, c)$$

$\implies \varphi$ is surjective

2 $\implies$ 3 $\psi : D \oplus C \longrightarrow B$

$$0 \to D \xrightarrow{f} B \underset{\curvearrowleft\, k}{\xrightarrow{g}} C \to 0$$

$\ker g = \mathrm{Im} f$ $\quad$ $g \circ k = id_C$

$$\psi((d, c)) = f(d) + k(c)$$

$\underline{\psi \text{ INJECTIVE:}}$ $f(d) = -k(c)$

$\Leftrightarrow$

$g(f(d)) = g(-k(c))$ $\leftarrow 0$

$\text{Im} f = \ker g$ $\quad \underbrace{g(-k(c))}_{-\text{id}_C} \Longleftrightarrow c = 0$

$\Longleftrightarrow d = 0$

$\underline{\psi \text{ SURJECTIVE:}}$ $\forall b \in B$

$$b = \underbrace{b - k(g(b))}_{\in \ker g} + \underbrace{k(g(b))}_{\in \text{Im} k}$$

$$g(b) - \text{id}_C\, g(b) = 0$$

$\ker g = \text{Im} f \Longrightarrow \exists\, d \in D$ s.t.

$f(d) = b - k(g(b))$

$\psi((d, g(b))) = b \Longrightarrow \psi$ surjective $\square$

ex. non-splitting exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot 1} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$\mathbb{Z} \not\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

§ Chain complexes

Def: • a chain complex $\mathcal{C}$ is a collection $\{(C_q, \partial_q)\}_{q\in\mathbb{Z}}$ where

C_q is an A-module

$\partial_q : C_q \longrightarrow C_{q-1}$ Boundary operator

s.t. $\partial_{q-1} \circ \partial_q = 0 \quad \forall q \in \mathbb{Z}$.

• The subgroup $Z_q^A(C) = \ker \partial_q \subseteq C_q$ is called group of q-cycles

• The subgroup $B_q^A(C) = \operatorname{Im} \partial_{q+1} \subseteq C_q$ is called of q-boundaries

$$\boxed{\partial_q \circ \partial_{q+1} = 0 \implies B_q^A(C) \subseteq Z_q^A(C)} \quad \forall q$$

quotient group

$$\boxed{H_q(C;A) := \frac{Z_q^A(C)}{B_q^A(C)}} \quad \forall q$$

q-th homology group of the complex C

Def: $C = \{(C_q, \partial_q)\}_{q\in\mathbb{Z}}$ $\quad C' = \{(C'_q, \partial'_q)\}_{q\in\mathbb{Z}}$

two chain complexes

A morphism $f: C \to C'$ is a collection

$\{f_q\}_{q\in\mathbb{Z}}$ $\quad f_q: C_q \to C'_q$ commuting

with the boundary operators

i.e. $\partial'_q \circ f_q = f_{q-1} \circ \partial_q \quad \forall q$

$$\begin{tikzcd}
\cdots C_q \arrow[r, "\partial_q"] \arrow[d, "f_q"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{q-1} \arrow[r, "\partial_{q-1}"] \arrow[d, "f_{q-1}"] \arrow[dr, phantom, "\circlearrowleft"] & C_{q-2} \cdots \arrow[d, "f_{q-2}"] \\
\cdots C'_q \arrow[r, "\partial'_q"] & C'_{q-1} \arrow[r, "\partial'_{q-1}"] & C'_{q-2} \cdots
\end{tikzcd}$$

RK: $f: C \to C'$ sends cycles to cycles and boundaries to boundaries, i.e.

$$\boxed{\begin{aligned} f_q(Z_q^{A}(C)) &\subseteq Z_q(C') \\ f_q(B_q^{A}(C)) &\subseteq B_q(C') \end{aligned}}$$

$\exists y \longmapsto x \in \operatorname{Im}\partial_q = B_{q-1}(C) \subset \ker\partial_{q-1} = Z_{q-1}(C)$

$$\begin{tikzcd} C_q \arrow[r, "\partial_q"] \arrow[d, "f_q"'] & C_{q-1} \arrow[r, "\partial_{q-1}"] \arrow[d, "f_{q-1}"] & C_{q-2} \arrow[d, "f_{q-2}"] \cdots \\ C'_q \arrow[r, "\partial'_q"] & C'_{q-1} \arrow[r, "\partial'_{q-1}"] & C'_{q-2} \cdots \end{tikzcd}$$

$z \in Z_q = \ker\partial_q \longmapsto 0$

x

0

$f_q(z) \longmapsto 0$

$\Rightarrow f_q(z) \in Z_q(C') = \ker\partial'_q$

$f_{q-1}(x) = \partial'_q(f_q(y))$

$\cap$

$\operatorname{Im}\partial'_q$

$\|$

$B_{q-1}(C')$

Therefore one can define

$$\boxed{\begin{aligned} H_q(f) &: H_q(C;A) \longrightarrow H_q(C';A') \\ H_q(f)&([z]) = [f_q(z)] \end{aligned}}$$

In fact $f_q(z) \in Z^{A}_q(C)$ and

if $[z] = [\bar{z}] \Longrightarrow z = \bar{z} + \partial_{q+1}(\sigma)$

they differ by a boundary

$$\begin{aligned} f_q(z) &= f_q(\bar{z}) + f_q(\partial_{q+1}(\sigma)) \\ &= f_q(\bar{z}) + \partial'_{q+1}(f_{q+1}(\sigma)) \end{aligned}$$

$$\Longrightarrow \quad [f_q(z)] = [f_q(\bar{z}) + 0]$$

Notation: if $A = \mathbb{Z}$, we drop the heavy notation

Recall that if $A = \mathbb{Z}$, you can just consider to work with abelian groups and group homomorphisms.

Many books present singular homology setting $A = \mathbb{Z}$.

Therefore

$$H_q(C) \cong H_q(C; \mathbb{Z})$$

We are mostly interested in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ for the purposes of this course.

§ Singular homology

We want to associate to a topological space a chain complex which reflects its geometry and, thereafter, we want to consider the homology group associated to such chain complex.

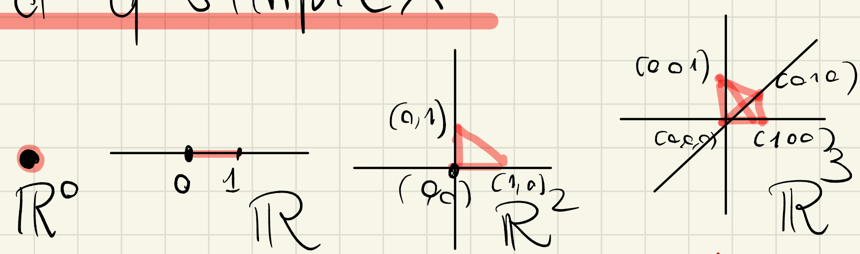
<u>Def</u>: Let $e_i = (0 \dots \underset{\substack{\uparrow \\ i\text{-th place}}}{1} \dots 0) \in \mathbb{R}^q \quad \forall i = 1 \dots q$

$$e_0 = (0 \dots \dots 0)$$

$$\Delta_q = \left\{ \sum_{i=0}^{q} \lambda_i e_i \ : \ \lambda_i \geq 0 \quad \sum \lambda_i = 1 \right\}$$

standard q-simplex

<u>example</u>:



$\Delta_0 \qquad \Delta_1 \qquad \Delta_2 \qquad \Delta_3$

Def: X topological space

- singular q-simplex is a continous map $\sigma : \Delta_q \longrightarrow X$
- support of σ, denoted with $|\sigma|$, is $\sigma(\Delta_q) \subset X$

Def: X topological space

$\forall q \geq 0 \qquad S_q^A(X) = \boxed{\text{free } A\text{-module generated by the singular } q\text{-simplexes of } X}$

IF $q<0$ set $S_q^A(X) = \{\underline{0}\}$

Such are called q-singular chains of X

We want to construct a chain complex $S^A(X)$ whose are $S_q^A(X)$. Therefore we need to define boundary operators

$$\partial_q : S_q^A(X) \longrightarrow S_{q-1}^A(X) \quad \forall q$$

$$\begin{tikzcd}
\Delta_q \arrow[r, "?"] \arrow[d, "\sigma"'] & \Delta_{q-1} \arrow[d, "\tau"] \\
X \arrow[r] & X
\end{tikzcd}$$

$$\xrightarrow{\partial_q}$$

$$\boxed{q \leq 0 \quad \partial_q = 0}$$

$$\boxed{q \geq 1}$$

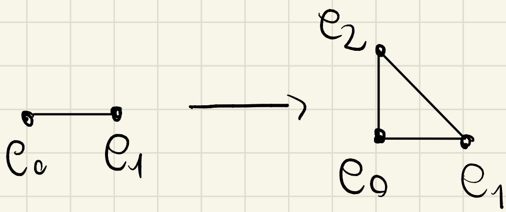
$\vdots$

Let us first define $F^i_{q+1} : \Delta_q \longrightarrow \Delta_{q+1}$

$i = 0, \dots, q$ starting from the vertices and extend it by linearity.

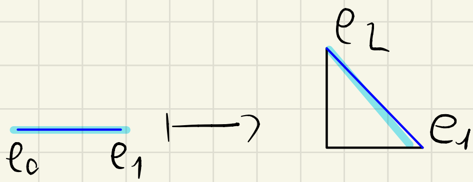
ex. $\boxed{F^{i}_{q+1}(e_j)} = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i \end{cases}$

FACE MAPS



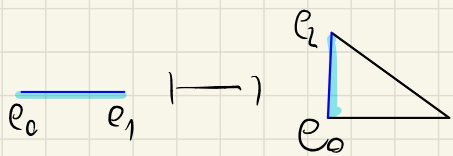
$F_1^0 : \quad e_0 \longmapsto e_1$

$\qquad e_1 \longmapsto e_2$



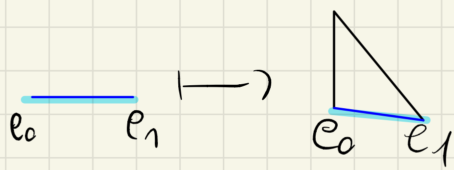
$F_1^1 : \quad e_0 \longmapsto e_0$

$\qquad e_1 \longmapsto e_2$



$F_1^2 : \quad e_0 \longmapsto e_0$

$\qquad e_1 \longmapsto e_1$



$$\boxed{\sigma : \Delta_q \longrightarrow X} \longmapsto \boxed{\sigma^{(i)} : \Delta_{q-1} \longrightarrow X}$$

$$\begin{tikzcd} \Delta_{q-1} \arrow[r, "F_q^i"] \arrow[rr, bend right] & \Delta_q \arrow[r, "\sigma"] & X \end{tikzcd}$$

$$\sigma^{(i)} := \sigma \circ F_q^i$$

Rk: $\sigma^{(i)}$ is the restriction of σ to the face of Δ_q which is opposite to the vertex e_i.

Lemma: Let $i \neq j$ between 0 and q. Denote with $\sigma^{(i,j)}$ the restriction of σ to the codimension 2 face of Δ_q not containing e_i and e_j. Then

$$F_q^i \circ F_{q-1}^j = \begin{cases} \sigma^{(i,j)} & i > j \\ \sigma^{(i,j+1)} & i \leq j \end{cases}$$

Proof:

$$F^{j}_{q-1}(e_k) = \begin{cases} e_k & k < j \\ e_{k+1} & k \geq j \end{cases}$$

$$F^{i}_{q}(e_k) = \begin{cases} e_k & k < i \\ e_{k+1} & k \geq i \end{cases}$$

$$F^{i}_{q}(e_{k+1}) = \begin{cases} e_{k+1} & k < i-1 \\ e_{k+2} & k \geq i-1 \end{cases}$$

$i \leq j$

$$F^{i}_{q} \circ F^{j}_{q-1}(e_k) = \begin{cases} e_k & k < i \\ e_{k+1} & i \leq k < j \\ e_{k+2} & k \geq j \end{cases},$$

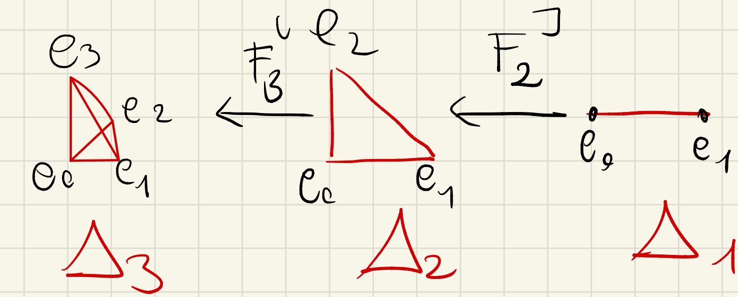
$i > j$

$$F^{i}_{q} \circ F^{j}_{q-1}(e_k) = \begin{cases} e_k & k < j \\ e_{k+1} & j \leq k < i-1 \\ e_{k+2} & k \geq i-1 \end{cases}$$

$$
\begin{tikzcd}
\Delta_{q-2} \arrow[r, "F^{j}_{q-1}"] \arrow[rrr, bend right] & \Delta_{q-1} \arrow[r, "F^{i}_{q}"] & \Delta_q \arrow[r, "\sigma"] & X
\end{tikzcd}
$$

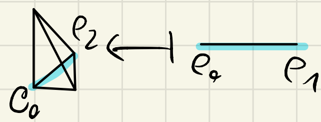
1 $\sigma^{(i,j+1)}$ 2 $\sigma^{(i,j)}$

example:



$i \leq j$

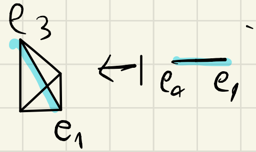
$$F_3^1 \circ F_2^2 (e_1) = e_2$$



$$F_3^1 \circ F_2^2 (e_0) = e_0$$

$i > j$

$$F_3^2 \circ F_2^0 (e_1) = e_3$$



$$F_3^2 \circ F_2^0 (e_0) = e_1$$

Boundary operators

Given a a singular q-complex σ,
define

$$\partial_q : \overset{A}{S_q}(X) \longrightarrow \overset{A}{S_{q-1}}(X)$$

$$\boxed{\partial_q(\sigma) = \sum (-1)^i \sigma^{(i)}}$$

and extend it by linearity

$$\partial_q\left(\sum m_i \sigma_i\right) = \sum m_i \partial_q(\sigma_i)$$

Lemma: ∂_q are such that $\partial_{q-1} \circ \partial_q = 0$

Proof: it is enough to prove this on the singular complexes.
Let $\sigma : \Delta_q \longrightarrow X$. Let us use the previous lemma

$$(\partial_{q-1} \circ \partial_q)(\sigma) = \partial_{q-1} \circ \left(\sum_{i=0}^{q} (-1)^i \underbrace{\sigma \circ F_q^i}_{\sigma^{(i)}} \right)$$

$$= \sum_{i=0}^{q} (-1)^i \partial_{q-1}(\sigma \circ F_q^i)$$

$$= \sum_{i=0}^{q} (-1)^i \sum_{j=0}^{q-1} (-1)^j (\sigma \circ F_q^i) \circ F_q^j$$

$$= \sum_{i \leq j} (-1)^{i+j} \sigma^{(i,j+1)} + \sum_{i > j} (-1)^{i+j} \sigma^{(i,j)}$$

$$\boxed{\tilde{j} = j+1}$$

$$= \sum_{i < j} (-1)^{i+j-1} \sigma^{(i,j)} + \sum_{i > j} (-1)^{i+j} \sigma^{(i,j)}$$

$$= 0 \qquad \square$$

<u>Def</u>: $\boxed{Z_q^A(X)} := \ker \partial_q \subset S_q(X)$

group of singular q-cycles

$\boxed{B_q^A(X)} := \operatorname{Im} \partial_{q+1} \subset S_q(X)$

group of singular q-boundaries

By the previous lemma $\partial_q \circ \partial_{q+1} = 0$

$$\Rightarrow \boxed{B_q^A(X) \subseteq Z_q^A(X)}$$

Therefore we ~~can~~ consider the quotient group

$$\boxed{H_q(X,A) = \frac{Z_q^A(X)}{B_q^A(X)}}$$

Rk: the homology measures how much the sequence is not exact

Remark: Since a singular simplex is a continous map and Δ_q is path connected, if $\{X_\gamma\}_{\gamma\in\Gamma}$ are the path connected components of X, then

$$S_q^A(X) \simeq \bigoplus_{\gamma\in\Gamma} S_q^A(X_\gamma) \quad \text{and}$$

$$\partial_q\left(S_q^A(X_\gamma)\right) \subset S_{q-1}^A(X_\gamma)$$

and

$$H_q(X;A) \simeq \bigoplus_{\gamma\in\Gamma} H_q(X_\gamma;A)$$

Examples: $X=\{x\}$ $A=\mathbb{Z}$

$\forall q\geq 0$ $\exists!$ continous map

$\sigma_q : \Delta_q \longrightarrow \{x\}$, which is the constant map.

So $S_q(X) = A\langle \sigma_q \rangle \quad \forall q \geq 0$

$$\partial_q : \mathbb{Z}\langle \sigma_q \rangle \longrightarrow \mathbb{Z}\langle \sigma_{q-1} \rangle$$

$$\mathbb{Z}\langle \sigma_q \rangle \cong \mathbb{Z} \qquad \mathbb{Z}\langle \sigma_{q-1} \rangle \cong \mathbb{Z}$$

$$\sigma_q \longmapsto \sum_{i=0}^{q} (-1)^i \sigma^{(i)} = \sum_{i=0}^{q} (-1)^i \sigma_{q-1} =$$

$$\partial_q(\sigma_q) = \begin{cases} 0 & q \text{ odd} \\ \sigma_{q-1} & q \text{ even}, \ q \neq 0 \end{cases}$$

$q = 0$

$$0 \xleftarrow{\partial_0} \mathbb{Z}\langle \sigma \rangle$$

$\Rightarrow \partial_0 = 0$

$$0 \xleftarrow{\partial_0} \mathbb{Z} \xleftarrow[0]{\partial_1} \mathbb{Z} \xleftarrow{\partial_2} \mathbb{Z} \xleftarrow[0]{\partial_3} \mathbb{Z} \xleftarrow{\partial_4} \cdots$$

$s \neq 0$

$$\operatorname{Ker} \partial_{2s+1} \cong \mathbb{Z} \qquad \operatorname{Im} \partial_{2s+1} = 0$$

$$\operatorname{Im} \partial_{2s+2} \cong \mathbb{Z} \qquad \operatorname{Ker} \partial_{2s} = 0$$

$$H_{2s+1}(\{x\};\mathbb{Z}) \cong \mathbb{Z}/\mathbb{Z} = 0$$

$$H_{2s}(\{x\};\mathbb{Z}) \cong 0$$

$q = 0$

$$\frac{\overbrace{\operatorname{Ker} \partial_0}^{\mathbb{Z}}}{\underbrace{\operatorname{Im} \partial_1}_{0}} \cong \mathbb{Z} \implies \begin{cases} H_0(X;\mathbb{Z}) \cong \mathbb{Z} \\ H_i(X;\mathbb{Z}) \cong 0 \quad i > 0 \end{cases}$$

Def: Let $f: X \to Y$ be a continuous map ① Let's define the induced morphism on the singular complexes

$S(f): S^A(X) \to S^A(Y)$:

$\forall q \quad S_q(f): S_q^A(X) \to S_q^A(Y)$

$$\sum m_i \sigma_i \mapsto \sum m_i (f \circ \sigma_i)$$

$$\Delta_q \xrightarrow{\sigma_i} X \xrightarrow{f} Y$$

Such morphisms commute with the boundary operators.

② Let's define the induced morphisms in homology $H_q(f): H_q(X;A) \to H_q(Y;A)$

$$H_q(f)\left(\left[\sum m_i \sigma_i\right]\right) = \left[S_q(f)\left(\sum m_i \sigma_i\right)\right] = \left[\sum_i m_i (f \circ \sigma_i)\right]$$

FACT: (HOMOTOPY INVARIANCE)

Let $f, g : X \to Y$ homotopic continous map. Then $H_q(f) = H_q(g)$ $\forall q \in \mathbb{Z}$

COROLLARY: $f : X \to Y$ homotopy equivalence then $H_q(f)$ is an isomorphism $\forall q \in \mathbb{Z}$

i.e. $H_q(X;A) \simeq H_q(Y;A)$.

example: any contractible topol. space X has only one non-trivial homology group $H_0(X;A) \simeq A$.

$\mathbb{R}^n$ is contractible $H_0(\mathbb{R}^n; A) = A$ $H_i = 0$ $\forall i \geq 0$

We would like to have a fast way to compute homology of given topological space, therefore we will introduce some tools.

Firstly, Mayer-Vietoris theorem will allow us to compute $H_q(X)$ from the homology of subspaces of X and their intersection.

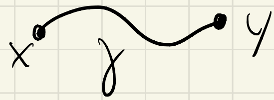
Proposition: if X is a topological space which is path connected then
$H_0(X;A) \simeq A$.

Proof: $S_0(X)$ 0-chain

$\forall x \in X \quad \exists\, \sigma_x : \Delta_0 \longrightarrow X$

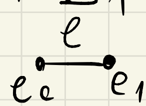
$\bullet \longmapsto x$

Given any two pts $x, y \in X$

$\exists$ path γ  since X is path-connected.

$S_1(X)$ 1-chain

$\forall x, y \quad \exists\, \sigma_\gamma : \Delta_1 \longrightarrow X$



$e_0 \longmapsto x$

$e_1 \longmapsto y$

$e \longmapsto \gamma$

Therefore any two points differ by a boundary

$\Longrightarrow H_0(X;A) = A\langle [x] \rangle$

Corollary: $X = \bigsqcup_{i \in I} X_i \quad X_i$ path-connected

$\Longrightarrow H_0(X;A) = \bigoplus_{i \in I} A$

§ Mayer-Vietoris theorem

Def: a sequence of chain complexes

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$ is said to be

EXACT if $\operatorname{Im} f = \ker g$ i.e

$$\operatorname{Im} f_q = \ker g_q \quad \forall q \in \mathbb{Z}.$$

PROPOSITION : (Long exact sequence)

Let $0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ a short exact sequence of chain complexes.

Then there exist homomorphisms

$$\alpha_q : H_q(C'') \longrightarrow H_{q-1}(C') \text{ s.t.}$$

$$\cdots \xrightarrow{H_q(f)} H_q(C) \xrightarrow{H_q(g)} H_q(C'') \longrightarrow H_{q-1}(C') \to \cdots$$

is a long exact sequence.

PROOF: let's consider the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d, "\partial'_{q+1}"'] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d, "\partial_{q+1}"'] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"] & 0 \\
0 \arrow[r] & C'_q \arrow[r, hook, "f_q"] \arrow[d, "\partial'_q"'] & C_q \arrow[r, two heads, "g_q"] \arrow[d, "\partial_q"'] & C''_q \arrow[r] \arrow[d, "\partial''_q"] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d, "\partial'_{q-1}"'] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d, "\partial_{q-1}"'] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}
$$

We want to define a homom.

$$\alpha_q : H_q(C'') \longrightarrow H_{q-1}(C')$$

$$\Psi$$

$[z''_q]$ where $z''_q \in Z''_q \subseteq C''_q$

g_q surjective $\forall q$ $\longrightarrow\!\!\!\!\to$

f_q injective $\forall q$ $\hookrightarrow$

LET US DEFINE α_q

$$
\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d, "\partial'_{q+1}"'] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d, "\partial_{q+1}"'] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"] & 0 \\
0 \arrow[r] & C'_q \arrow[r, hook, "f_q"] \arrow[d, "\partial'_q"'] & C_q \arrow[r, two heads, "g_q"] \arrow[d, "\partial_q"'] & C''_q \arrow[r] \arrow[d, "\partial''_q"] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d, "\partial'_{q-1}"'] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d, "\partial_{q-1}"'] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}
$$

$\exists\; c_q \mapsto z''_q$

$\exists\; c'_{q-1} \mapsto \partial_q(c_q)$

1. g_q surjective $\exists\, c_q \in C_q : g_q(c_q) = z''_q$

2. $g_{q-1}(\partial_q(c_q)) = \partial''_q(\underbrace{z''_q}_{\in \ker \partial''_q}) = 0$

$\Rightarrow \partial_q(c_q) \in \ker g_{q-1} = \operatorname{Im}(f_{q-1})$

3 $\exists\ c'_{q-1} \in C'_{q-1}$ s.t.

$$f_{q-1}(c'_{q-1}) = \partial_q(c_q)$$

Moreovere,

4 $f_{q-2}(\partial'_{q-1}(c'_{q-1})) = \partial_{q-1}(\partial_q(c_q)) = 0$

$\Longrightarrow$ $\partial'_{q-1}(c'_{q-1}) = 0 \Longrightarrow$

f_{q-2} is injective

5) $c'_{q-1} \in Z'_{q-1} \subset C'_{q-1}$

and we can consider $[c'_{q-1}]$ in $H_{q-1}(C')$.

Put $\alpha_q([z''_q]) = [c'_{q-1}]$

DEFINITION OF α_q

$$\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d] & C''_{q+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{q} \arrow[r, hook, "f_{q}"] \arrow[d] & C_{q} \arrow[r, two heads, "g_{q}"] \arrow[d] & C''_{q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d] & C''_{q-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}$$

LEFT TO CHECK:

1. α_q well defined
2. α_q homomorphism
3. $H_q(C^\bullet)$ is an exact sequence

1) ∂_q well defined

First, let us show that the definition does not depend on the choice of c_q, pre-image of z_q'' via g_q.

Let $\bar{c}_q$ be another pre-image of z_q'' via g_q. Then $g_q(\bar{c}_q - c_q) = 0$

$\Rightarrow \bar{c}_q - c_q \in \ker g_q = \operatorname{Im} f_q$

$\Rightarrow \exists\ \bar{c}'_q \in C'_q$ s.t.

$$\bar{c}_q = c_q + f_q(\bar{c}'_q) \Longrightarrow$$

$$\partial_q(\bar{c}_q) - \partial_q(c_q) = \partial_q(f_q(\bar{c}'_q)) \underset{\text{COMMUT. DIAGRAM.}}{=} f_{q-1}(\partial'_{q-1}(\bar{c}'_q))$$

$\Rightarrow$ therefore the element in C_{q-1} defined starting from $\bar{c}_q$ differs from that defined from c_q by a boundary $\partial'_{q-1}(\bar{c}_q)$ and, therefore, such elements represent the same class in $H_q(C')$.

$$\bar{c}_q' \xleftarrow{\;2\;} \bar{c}_q - c_q \xmapsto{\;1\;} 0$$

$$\begin{tikzcd}
0 \arrow[r] & C'_q \arrow[r, hook, "f_q"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C_q \arrow[r, two heads, "g_q"] \arrow[d] & C''_q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"'] & C_{q-1} \arrow[r, two heads, "g_{q-1}"'] & C''_{q-1} \arrow[r] & 0
\end{tikzcd}$$

3

$\partial'_q(\bar{c}'_q)$ $\quad$ $\partial_q(\bar{c}_q) - \partial_q(c_q)$ $\quad$ 4

Let us show that the definition does not depend on the choice of the representative of the class $[z''_q]$. Let $\bar{z}''_q$ s.t. $[\bar{z}''_q] = [z''_q]$.

We have by definition that

$\bar{z}''_q - z''_q = \partial''_{q+1}(c''_{q+1})$; since g_{q+1} is surjective $\exists\, c_{q+1} \in C_{q+1}$ s.t. $g_{q+1}(c_{q+1}) = c''_{q+1}$

So $\bar{z}''_q - z''_q = \partial''_{q+1}\left(g_{q+1}(c_{q+1})\right)$

$= g_q\left(\partial_{q+1}(c_{q+1})\right)$

Now, since $\partial_q\left(\partial_{q+1}(c_{q+1})\right) = 0$

it follows that the definition of δ_q does not depend on the chosen representative.

c_{q+1} $\qquad$ c''_{q+1}

$\bar{z}''_q - z''_q$

0

$$\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, hook, "f_{q+1}"] \arrow[d, "\partial'_{q+1}"] & C_{q+1} \arrow[r, two heads, "g_{q+1}"] \arrow[d, "\partial_{q+1}"] & C''_{q+1} \arrow[r] \arrow[d, "\partial''_{q+1}"] & 0 \\
0 \arrow[r] & C'_q \arrow[r, hook, "f_q"] \arrow[d, "\partial'_q"] & C_q \arrow[r, two heads, "g_q"] \arrow[d, "\partial_q"] & C''_q \arrow[r] \arrow[d, "\partial''_q"] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, hook, "f_{q-1}"] \arrow[d, "\partial'_{q-1}"] & C_{q-1} \arrow[r, two heads, "g_{q-1}"] \arrow[d, "\partial_{q-1}"] & C''_{q-1} \arrow[r] \arrow[d, "\partial''_{q-1}"] & 0 \\
0 \arrow[r] & C'_{q-2} \arrow[r, hook, "f_{q-2}"] & C_{q-2} \arrow[r, two heads, "g_{q-2}"] & C''_{q-2} \arrow[r] & 0
\end{tikzcd}$$