Real algebraic curves in real del Pezzo surfaces

Matilde Manzaroli

université
PARIS-SACLAY

NNT : 2019SACLX017

Thèse de doctorat

Real algebraic curves on real minimal del Pezzo surfaces

Thèse de doctorat de l'Université Paris-Saclay
préparée à l'École Polytechnique

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques fondamentales

Thèse présentée et soutenue à Palaiseau, le 28/06/2019, par

MANZAROLI MATILDE

Composition du Jury :

Charles FAVRE Professeur, Centre Mathématiques Laurent Schwartz - École polytechnique	Président
Erwan BRUGALLÉ Professeur, Université de Nantes - Laboratoire de Mathématiques Jean Leray	Directeur de thèse
Penka GEORGIEVA Maître de Conférence, Institut de Mathématiques de Jussieu - Paris Rive Gauche, Sorbonne Université	Examinateur
Ilia ITENBERG Professeur, Institut de Mathématiques de Jussieu - Paris Rive Gauche, Sorbonne Université	Examinateur
Grigory MIKHALKIN Professeur, Section de Mathématiques, Université de Genève	Examinateur
Eugenii SHUSTIN Professeur, School of Mathematical Sciences at the Tel-Aviv University	Rapporteur
Jean-Yves WELSCHINGER Professeur, Institut Camille Jordan - Université Lyon1	Rapporteur

Remerciements

I am grateful to many people that I have met during the last four years. In particular, I want to say thank you to

- My advisor Erwan Brugallé for his support, many fruitful discussions, his believing in me and his effort to make me discover many aspects of research and life. A special thought for Assia and Inés;

- The referees of my PhD thesis Eugenii Shustin and Jean-Yves Welschinger for their time, very useful remarks and questions for future directions;

- All the members of the jury to be here: Erwan Brugallé, Charles Favre, Penka Georgieva, Ilia Itenberg, Grigory Mikhalkin, Eugenii Shustin and Jean-Yves Welschinger;

- Ilia who was very supportive in those years. Moreover, everybody in Jussieu has meant a lot for me; in particular, thank you to Penka, Alexandru, Thomas, Charles, Lorenzo, Matteo, Vincent, Johannes J.;

- The "real and tropical" group to have made me feel part of something really amazing; in particular, Benoit, Kristin, Arthur, Lucia, Erwan, Andrés, Johannes R., Johannes J., Cristhian, Boulos, Grhisha, Eugenii, Sergei, Frédéric, Patrick, Ilia, Anh, Penka, Michele, Antonio, Jean-Yves, Sergey, Alex, Thomas, Charles;

- The two laboratories that have hosted me: Centre de Mathématiques Laurent Schwartz, École Poltechnique and Laboratoire de Mathématiques Jean-Leray, Université de Nantes; especially, the wonderful secretaries of CMLS and all the Phd students. A special thought to Bac, Nicolas, Fabio, Jacek, Rita, Marco, Paolo, my *little sister* Anh, Fabien, and to all my "co-bureaux" Ivan, Vincent, German and Caroline;

- Andrés to help me in my understanding of dessins d'enfant and everything else!

- People and programs which have helped me to travel for my mathematical achievements; in particular, Erwan, Kristin, everybody who has invited me to give a talk, ECOS NORD program, EDMH, CIMPA, CMLS, Laboratoire de Mathématiques Jean-Leray;

- all the people of UNAM (Universidad Nacional Autónoma de México) that I had the pleasure to meet in Cuernavaca and Mexico City; especially, a big thanks to *my roommie* Lucia, Aubin, Cristhian and Jawad.

A super hug to: Remi who is a very special and crazy friend to me, Vlad, Livia, Egor, Alex, Carletto, Michele, Sebastian, Arindam, Alexandre, Elena, Nic, all my *sud-west* friends, Francesca, SH, KMD, TKD, Rosamaria, Pollo, Raphaël, Vince, Marta, Ari.

I want to thank all my family. In particular, I am really grateful to my parents Dario e Patrizia who have always been supportive during every my little step and to my "french" family who has been super charming and welcoming to me. A special thanks to Anna for her energy.
Last, but not least, thank you very much to my "almost non-mathematical" boyfriend Emiliano who has very particular sense of humor, but he's the best and the most wonderful person ever. He has always done his best to be there for me and to make me spent some crazy and amazing adventures.

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Chapter 1

Introduction

1.1 Motivation

L'étude topologique des variétés algébriques réelles remonte au moins aux travaux de Harnack, Klein, et Hilbert au 19ème siècle ([Har76], [Hil02], [Kle22]); en particulier, la classification des types d'isotopie réalisés par les courbes algébriques réelles d'un degré fixé dans $\mathbb{R}P^2$ est un sujet qui a connu un essor considérable jusqu'à aujourd'hui. En revanche, en dehors des études concernants les surfaces de Hirzebruch et les surfaces de degré au plus 3 dans $\mathbb{R}P^3$ (pour avoir un aperçu général voir [DIK00]), à peu près rien n'est connu dans le cas de surfaces ambiantes plus générales. Cela est dû en particulier au fait que les variétés construites en utilisant le "patchwork" sont des hypersurfaces de variétés toriques ([Vir84a], [Vir84b], [Vir89]). Or, il existe de nombreuses autre surfaces algébriques réelles. Parmi celles-ci se trouvent les surfaces rationnelles réelles, et plus particulièrement les surfaces $\mathbb{R}$-minimales ou minimales (voir Section 1.3). Notons au passage que la classification des surfaces rationnelles réelles minimales est beaucoup plus riche que dans le cas complexe. En particulier, la partie réelle d'une surface rationnelle réelle n'est pas nécessairement connexe. Dans cette thèse, on élargit l'étude des types d'isotopie réalisés par les courbes algébriques réelles aux surfaces réelles minimales de del Pezzo de degré 1 et 2 pour lesquelles le groupe de Picard réel est engendré par la classe anti-canonique. La première surface est un revêtement double de $\mathbb{C}P^2$ ramifié le long d'une quartique réelle maximale, et sa partie réelle est composée de quatre sphères. La seconde surface est un revêtement double du cône quadratique dans $\mathbb{C}P^3$ ramifié le long d'une section cubique réelle maximale, et sa partie réelle est composée de quatre sphères et d'un plan projectif. Les méthodes employées combineront l'application de résultats classiques, l'application des invariants de type Welschinger sur les contraintes satisfaites par ces types d'isotopies, et l'utilisation de méthodes de construction de dégénérescence ayant récemment trouvé des applications en géométrie énumérative réelle.

1.2 Historique

Une variété algébrique réelle est une variété algébrique compacte complexe X munie d'une involution anti-holomorphe $\sigma : X \to X$, appelée structure réelle sur X. La partie réelle $\mathbb{R}X$ de (X, σ) est l'ensemble des points fixes de l'involution σ. Une question intéressante sur les variétés algébrique réelle concerne la topologie de leur partie réelle. L'un des premiers résultats significatifs est la classification des surfaces cubiques réelles donnée dans [Sch63]. Ensuite, dans [Zeu74], on trouve une étude des courbes algébriques réelles planes de degré 4 et de leurs bitangentes (qui équivaut à l'étude des surfaces de del

Pezzo réelles de degré 2). La première étude systématique sur la topologie des varétés algébriques réelles commence avec Harnack, Klein, Hilbert et Comesatti ([Har76], [Hil02], [Com13], [Com14], [Kle22]). Hilbert proposa dans la première partie de son 16ème problème de classifier les types d'isotopie des courbes algébriques réelles de degré 6 dans $\mathbb{R}P^2$, et les types topologiques des surfaces algébriques réelles de degré 4 dans $\mathbb{R}P^3$. La première classification fut achevée à la fin des années 60 par Gudkov ([Gud69]), et la deuxième par Kharlamov([Kha76], [Kha78]) une dizaine d'années plus tard. En général, il y a deux directions principales dans l'étude de ce sujet : la première consiste à obstruer la topologie des variétés algébriques réelles ; la seconde concerne la contruction des variétés algébriques réelles à topologie prescrite. À la suite de travaux initiés en particulier par Arnold et Rokhlin au début des années 70 ([Arn71],[Rok72],[Rok74],[Rok78],[Rok80]), de nombreuses et très générales obstructions sur la topologie des variétés algébriques réelles ont été mises à jour. En revanche, les méthodes de construction de telles variétés à topologie prescrite sont très longtemps restées relativement élémentaires, et ce même dans le cas des courbes planes. En 1979, Viro inventa une nouvelle méthode, dite de "patchwork", pour construire des hypersurfaces algébriques réelles dans des variétés toriques réelles ; voir Section 2.2. Cette technique révolutionna le domaine et permit en particulier à Viro de terminer la classification des courbes réelles de degré 7 dans $\mathbb{R}P^2$ ([Vir84a], [Vir84b]). Le patchwork a depuis connu de nombreuses généralisations et a été utilisé pour obtenir des résultats importants, citons par exemple les travaux sur les lissages de singularités dégénérées de courbes planes ([Shu98]). La méthode de Viro est toujours l'une des méthodes de construction les plus puissantes connues à ce jour, et dont l'importance a largement dépassé le cadre de la topologie des variétés algébriques réelles.
Dans le cas de la construction d'hypersurfaces algébriques réelles dans des variétés non-toriques ou dans des variétés toriques à structure réelle non compatible avec l'action du tore, la méthode de Viro n'est pas directement applicable. Une approche possible pour surmonter cette difficulté est d'essayer de dégénérer la variété ambiante à une variété réductible sur laquelle on peut utiliser des méthodes classiques de construction.

1.3 Généralités

On dit que une surface réelle (X, σ) est *rationnelle* si X est une surface birationallement equivalent à $\mathbb{C}P^2$. On dit que (X, σ) est $\mathbb{R}$-*minimale* ou *minimale* si toute fonction $f : X \to Y$ réelle holomorphe de degré 1 à une surface réelle (Y, τ) est un biholomorphisme. Autrement dit (X, σ) est $\mathbb{R}$-minimale si et seulement si il n'y a pas de (-1)-courbes réelles ou de paires de (-1)-courbes disjointes complexes σ-conjugués. Les surfaces algébriques réelles rationelles ont étés classifiées par Comessatti ([Com13], [Com14]). L'approche de Comessatti est de se ramener à la classification des surfaces algébriques réelles rationelles $\mathbb{R}$-minimales ; pour un aperçu général voir [Kol97], [DK02], [Man17, Chapitre 4]. Puisque on peut obtenir toute surface rationelle réelle comme une suite d'éclatements réels d'une surface rationelle réelle et minimale, la première étape pour étudier la topologie des courbes algébriques réelles sur les surfaces rationelles réelles est de l'étudier sur les surfaces rationelles réelles et minimales. Cela, en partie, motive le choix d'analyser dans cette thèse les surfaces de del Pezzo réelles minimales de degré 1 et 2.
Nous allons présenter quelques résultats généraux applicables pour la classification des courbes algébriques réelles. Un de ces résultats est *l'inegalité de Harnack-Klein*, montrée par Harnack dans [Har76] pour les courbes al-

gébriques réelles planes, et généralisée ensuite par Klein ([Kle22]). Elle donne une obstruction qui vaut pour toute courbe algébrique réelle compacte et non-singuliére.

Theorem 1.3.1 (Inegalité de Harnack-Klein). *Soit (A, σ) une courbe algébrique réelle et compacte, alors le nombre l des composants connexes de $\mathbb{R}A$ est borné par $g+1$, où $g = g(A)$ est le genre de A.*

Definition 1.3.2. *On dit que A est une M-courbe ou une courbe maximale, si $l = g+1$. Autrement, si $0 \leq l \leq g$, on dit que A est une $(M-i)$-courbe, où $i = g+1-l$.*

Example 1.3.3 (Application de Harnack-Klein). *Soit A une courbe algébrique réelle plane et non-singuliere de degré d. Alors,*

$$l \leq \frac{(d-1)(d-2)}{2} + 1.$$

L'objet d'étude de la topologie algébrique réelle a rapidement évolué plaçant la partie réelle de la courbe et sa position par rapport à sa complexification au centre de l'attention. Klein ([Kle22]) a introduit les définitions suivantes.

Definition 1.3.4. *Si $A \setminus \mathbb{R}A$ est connexe on dit que A est de type II ou non-séparante, autrement de type I ou séparante.*

Ce point de vue conduit, par exemple, à obtenir des informations sur le nombre l. En général, on sait que si A est maximale alors la courbe est de type I. En outre, si A est de type I alors l a la même parité de $g+1$. Rokhlin a également promu cette nouvelle approche et y a fortement contribué ([Rok72], [Rok74], [Rok78], [Rok80]) ; un exemple important de sa contribution est l'introduction et l'étude des *orientations complexes* d'une courbe algébrique réelle.

Definition 1.3.5. *Les deux moitiés de $A \setminus \mathbb{R}A$ induisent deux orientations opposées sur $\mathbb{R}A$ dites orientations complexes de la courbe.*

Ce changement de perspectives a permis un progrès remarquable dans l'étude de la topologie des courbes algébriques réelles et dans le raffinement de classifications.
Il existe d'autres moyens d'obstruer la topologie des courbes réelles. Certains d'entre eux sont basés sur la forme d'intersection qu'on peut définir sur les variétés C^∞ orientés. Citons, par exemple, les obstructions du type Bézout, les congruences d'Arnold ([Arn71]), les congruences de Rokhlin ([Rok72]) et ses généralisations ([GM77]).

1.4 Résultats

Le Chapitre 2 est consacré à introduire des outils déjà connus et des notations qu'on utilisera dans tous les autres chapitres. En particulier, on présente différents types de construction : le patchwork (Section 2.2.2), certaines de ses variantes (Sections 2.2.3.1, 2.2.4) et la méthode des dessins d'enfant (Section 2.4).
Les Chapitres 3, 4 et 5 sont dédiés aux résultats principaux de cette thèse. On s'intéresse d'abord à la classification des courbes algébriques réelles de bidegré $(5, 5)$ sur la quadrique ellipsoïde. Ensuite, le fil conducteur est l'étude de la topologie des courbes algébriques réelles dans les surfaces de del Pezzo minimales réelles de degré 1 et 2.

1.4.1 Chapitre 3: Quadrique ellipsoïde

Les isotopies rigides réalisées par les courbes algébriques réelles de bidegré (d, d) sur la quadrique ellipsoïde sont classifées pour tout $d < 5$ (voir [GS80],[Zvo91], [NS05a], [NS07],[NS05b], [DZ99], [Nik85], [Mik94] et, pour avoir un aperçu général, voir [DK00, Section 4.9]). Dans le Chapitre 3, on termine la classification, à homéomorphisme près, des courbes algébriques réelles de bidegré $(5, 5)$ sur la quadrique ellipsoïde (ce chapitre est sous le format d'un article dans [Man18]). En particulier, on démontre que les résultats précédents ([Zvo83], [Mik94, Theorem 1], [DK00, Proposition 4.9.2], [Ore07, Proposition 1.2]) sur les obtructions topologiques des courbes algébriques de bidegré $(5, 5)$ constituent un système complet d'obstructions ; en outre, nous finissons la réalisation des types topologiques, commencée par Mikhalkin ([Mik94]), par des courbes algébriques réelles maximales (Théorème 3.1.2). En plus, on réalise tous types topologiques par des courbes algébriques réelles séparantes et/ou non-séparantes (Théorèmes 3.1.3, 3.1.4 and 3.1.5).

Théorème 1.4.1. *Soit A une courbe algébrique réelle non-singulière non-séparante (resp. séparante) de bidegré $(5, 5)$ sur la quadrique ellipsoïde. Alors A réalise un type topologique parmi ceux non-interdits par les restrictions dans [Zvo83], [Mik94], [DK00] et [Ore07]. En plus, chacun de ces types topologiques non-interdits est réalisable par une courbe algébrique réelle non-singulière non-séparante (et/ou séparante) de bidegré $(5, 5)$ sur la quadrique ellipsoïde.*

La quadrique ellipsoïde est une surface algébrique torique équipée avec une structure réelle incompatible avec l'action du tore. Ne pouvant pas utiliser directement le patchwork sur la quadrique ellipsoïde, la stratégie principale de construction est basée sur la fait de se ramener à la construction de courbes algébriques réelles sur la deuxième surface de Hirzebruch Σ_2 en dégénérant la quadrique ellipsoïde au cône quadratique dans $\mathbb{C}P^3$ (Section 3.3).

1.4.2 Chapitre 4 et 5: Surfaces de del Pezzo réelles minimales de degré 1 et 2

La classification des courbes algébriques réelles sur les surfaces de del Pezzo réelles minimales est délicat car, en plus d'être des surfaces non-toriques, elles ont également une partie réelle non-connexe. En 1998, Mikhalkin ([Mik98]) fut le premier à faire face à la difficulté de classifier des courbes algébriques réelles sur des surfaces réelles à partie réelle non-connexe. En particulier, il a étudié la topologie de la partie réelle des intersections transverses des surfaces quadriques réelles avec les surfaces cubiques réelles en $\mathbb{C}P^3$. Par contre, l'avantage de travailler sur les surfaces de del Pezzo réelles minimales est que les courbes algébriques réelles réalisent en homologie un multiple entier positif de la classe anti-canonique.

Chapitre 4: Surfaces de del Pezzo réelles minimales de degré 1

Une surface de del Pezzo Y réelle minimale de degré 1 a la partie réelle composée de quatre sphères et d'un plan projectif. Le double du système anti-canonique présente Y comme revêtement double du cône quadratique dans $\mathbb{C}P^3$ ramifié le long d'une section cubique réelle maximale ; voir Fig. 1.1. Dans le Chapitre 4, on propose divers raffinements du problème de classification des courbes algébriques réelles sur Y (Definition 4.1.4) et on effectue ces classifications pour les courbes algébriques réelles de classe $kc_1(Y)$, pour

$k \leq 3$, où $c_1(Y)$ est la classe anti-canonique de Y (Théorème 4.1.6). On montre que pour ces classes, une restriction due à la forme d'intersection sur Y et l'inégalité de Harnack-Klein fournissent un système complet d'obstructions.

Théorème 1.4.2. *Soit A une courbe algébrique réelle non-singulière de classe $kc_1(Y)$, avec $k \in \{1, 2, 3\}$, dans Y. Alors A réalise un type topologique parmi ceux non-interdits par la forme d'intersection sur Y et l'inégalité de Harnack-Klein. En plus, chacun de ces types topologiques est réalisable par une courbe algébrique réelle non-singulière de classe $kc_1(Y)$ sur Y, pour $k \leq 3$.*

En plus, nous démontrons une variante du théorème de Bézout qui donne d'autres obstructions pour les courbes de classe $kc_1(Y)$, avec $k \geq 4$ (Proposition 4.2.3).

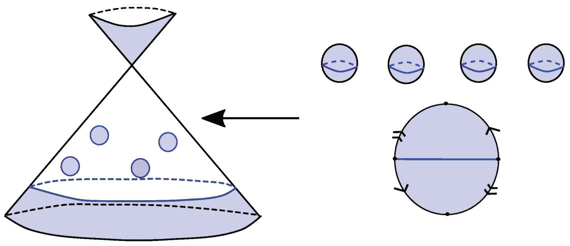


Figure 1.1: Action du double du système anti-canonique sur la partie réelle.

Chapitre 5: Surfaces de del Pezzo réelles minimales de degré 2 avec 4-spheres

Dans le Chapitre 5, on s'interesse aux surfaces de del Pezzo X de degré 2 réelles minimales et avec partie réelle composée de quatre sphères. Le système anti-canonique présente X comme un revêtement double de $\mathbb{C}P^2$ ramifié le long d'une quartique réelle dont la partie réelle a quatre composants connexes; voir Fig. 1.2. Ici le problème de classification des courbes algébriques réelles

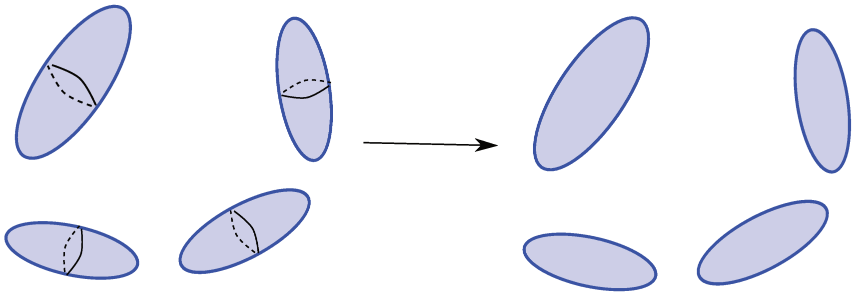


Figure 1.2: Action du système anti-canonique sur la partie réelle.

est plus simple à énoncér que dans le cas des surfaces de del Pezzo de degré 1. On effectue cette classification pour les courbes algébriques réelles de classes $kc_1(X)$, pour $k \leq 2$ (Proposition 5.1.6). Dans cette situation, on montre que pour ces classes, l'inégalité de Harnack Klein fournit un système complet d'obstructions.

Théorème 1.4.3. *Soit A une courbe algébrique réelle non-singulière de classe $kc_1(X)$, avec $k = 1, 2$, dans X. Alors A réalise un type topologique parmi ceux non-interdits par l'inégalité de Harnack-Klein. En plus, chacun de ces types topologiques est réalisable par une courbe algébrique réelle non-singulière de classe $kc_1(X)$ sur X, pour $k \leq 2$.*

On s'attaque ensuite au cas des courbes réelles maximales de classe $3c_1(X)$, qui se montre beaucoup plus délicat à étudier (Théorème 5.1.7). D'une part, aucune des obstructions classiques ne semble donner des résultats intéressants en dehors d'une variante du théorème de Bézout et de l'inégalité d'Harnack-Klein. D'autre part, les méthodes de constructions utilisées laissent ouverts la réalisabilité d'environ un tiers des types topologiques non-interdits. Parmi ces types topologiques, il en y a un qu'on arrive à réaliser par une courbe symplectique (Proposition 5.4.2).
Dans le Chapitre 5, on montre le résultat suivant.

Théorème 1.4.4 (Classe 3)**.** *Soit A une courbe algébrique réelle maximale non-singulière de classe $3c_1(X)$ dans X. Alors A réalise un type topologique parmi les 74 types topologiques non-interdits par une variante du théorème de Bézout et l'inégalité de Harnack-Klein. En plus 47 parmi ces 74 types topologiques sont réalisables par des courbes algébriques réelles non-singulières de classe $3c_1(X)$ sur X.*

La méthode principale de construction pour les courbes de classe $3c_1(X)$ est basée sur une construction de dégénérescence ayant récemment trouvé des applications en géométrie énumérative réelle (Proposition 5.4.7, Corollary 5.4.8) et l'application d'une variante du théorème de patchwork due à Shustin et Tyomkin (Section 2.2.4).
En plus, nous obtenons de nouvelles obstructions pour les courbes de classe $kc_1(X)$, avec $k \geq 4$, grâce aux invariants du type Welschinger (Proposition 5.2.2).

Chapter 2

Preliminaries

2.1 Encoding topological types

Let X be a real algebraic surface equipped with a real structure $\sigma : X \to X$. Let $\sigma_* : H_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z})$ be the group homomorphism induced by the real structure σ on X, and let $H_2^-(X;\mathbb{Z})$ be the (-1)-eigenspace of σ_*. In the following, fixed a homology class $\alpha \in H_2^-(X;\mathbb{Z})$, we are interested in the classification of the topological types of the pair $(\mathbb{R}X, \mathbb{R}A)$ up to homeomorphism of $\mathbb{R}X$, where $A \subset X$ is a non-singular real algebraic curve realizing α in $H_2(X;\mathbb{Z})$. The real part of A is homeomorphic to a union of circles embedded in $\mathbb{R}X$. It follows that, depending on the first homology group $H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$, a circle can be embedded in $\mathbb{R}X$ in different ways. For the purpose of this thesis, we only need to explain how to encode the embedding of a given collection $\bigsqcup_{i=1,..l} B_i$ of l disjoint circles in $\mathbb{R}P^2$ and in S^2. A circle embedded in S^2 is called *oval*. On the other hand, a circle can be embedded in $\mathbb{R}P^2$ in two ways: if it realizes the trivial-class in $H_1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$, it is called *oval*; otherwise it is called *pseudo-line*.

Let us call *oval* a circle embedded in $\mathbb{R}^2$. Let us consider a collection $\bigsqcup_{i=1,..l} B_i$ of circles embedded in $\mathbb{R}^2$, resp. in $\mathbb{R}P^2$. An oval B_i in $\mathbb{R}^2$, resp. in $\mathbb{R}P^2$, separates two disjoint non-homeomorphic connected components: the connected component homeomorphic to a disk is called *interior* of the oval; the other one is called *exterior* of the oval. For each pair of ovals, if one is in the interior of the other we speak about an *injective pair*, otherwise of a *non-injective pair*. We shall adopt the following notation to encode a given topological pair $(\mathbb{R}^2, \bigsqcup_{i=1,..l} B_i)$, resp. $(\mathbb{R}P^2, \bigsqcup_{i=1,..l} B_i)$.

An empty union of ovals is denoted by 0. We say that a union of l ovals realizes l if there are no injective pairs. The symbol $\langle \mathcal{S} \rangle$ denotes the disjoint union of a collection of ovals realizing $\mathcal{S}$, and an oval forming an injective pair with each oval of the collection. Finally, the disjoint union of any two collections of ovals, realizing respectively $\mathcal{S}'$ and $\mathcal{S}''$ in $\mathbb{R}^2$ (resp. $\mathbb{R}P^2$), is denoted by $\mathcal{S}' \sqcup \mathcal{S}''$ if none of the ovals of one collection forms an injective pair with the ovals of the other one. Moreover, a pseudo-line in $\mathbb{R}P^2$ is denoted by $\mathcal{J}$.

Since $\mathbb{R}^2$ is homeomorphic to S^2 deprived of a point, we say that the pair $(S^2, \bigsqcup_{i=1,..,l} B_i)$ realizes $\mathcal{S}$ if there exists a point $p \in S^2 \setminus \bigsqcup_{i=1,..,l} B_i$ such that $(S^2 \setminus \{p\}, \bigsqcup_{i=1,..,l} B_i)$ realizes $\mathcal{S}$.

As example, we have depicted in a) of Fig. 2.1 an arrangement of 8 ovals in S^2 projected on a plane from some point $p \in S^2$. The pair $(S^2, \bigsqcup_{i=1,..,8} B_i)$ realizes $1 \sqcup \langle 2 \rangle \sqcup \langle 3 \rangle$. While, in b) of Fig. 2.1 we have an example of 6 circles embedded in $\mathbb{R}P^2$ such that the arrangement of $(\mathbb{R}P^2, \bigsqcup_{i=1,..,6} B_i)$ realizes $\mathcal{J} \sqcup \langle\langle 1 \rangle\rangle \sqcup \langle 1 \rangle$.

Definition 2.1.1. *Let X be a real algebraic surface and $A \subset X$ a real curve. We say that A has real scheme $\mathcal{S}$ if the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes $\mathcal{S}$.*

Finally, in the following chapters, we need some definitions for particular collections of ovals in $\mathbb{R}P^2$ and S^2.

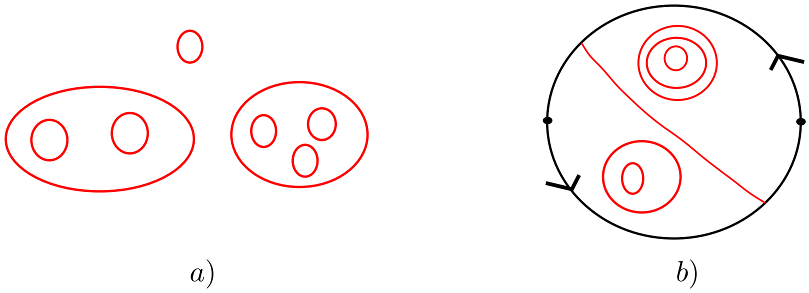


Figure 2.1: Example of arrangements of embedded circles in S^2 and $\mathbb{R}P^2$.

Definition 2.1.2. *A collection of h ovals in $\mathbb{R}P^2$ is called a nest of depth h if any two ovals of the collection form an injective pair. Let N_1 and N_2 be two nests of depth i_1 and i_2 in $\mathbb{R}P^2$. We say that the nests are disjoint if each pair of ovals, composed by an oval of N_1 and an oval of N_2, is non-injective.*

Definition 2.1.3. *A collection N_h of h ovals in S^2 is a nest if each connected component of $S^2 \setminus N_h$ is either a disk or an annulus.*
Let N_{i_k} be k nests of depth i_k in S^2, with $k \geq 3$. We say that the nests are disjoint if a disk of $S^2 \setminus N_{i_j}$ contains all other $k-1$ nests, for all $j \in \{1, .., k\}$.

2.2 Patchworking

The Viro's patchworking method was developed in the 70's by Viro; it turned out to be one of the most powerful method to construct real algebraic hypersurfaces with prescribed topology in real algebraic toric varieties; for example, Viro used it to classify non-singular real algebraic curves of degree 7 in $\mathbb{R}P^2$ ([Vir84a]).

2.2.1 Toric varieties

First all all, let us give some definitions about toric varieties associated to a convex polytope and about subdivisions of polytopes; more details about these subjects can be found in [Ful93], [GKZ94].

Definition 2.2.1. *A toric variety is an irreducible normal complex algebraic variety equipped with an action of an algebraic torus $(\mathbb{C}^*)^n$ having an open dense orbit.*

Definition 2.2.2. *An integer convex polytope in $\mathbb{R}^n$ is the convex hull of a finite subset of $\mathbb{Z}^n$.*

For $z = (z_1, .., z_n) \in (\mathbb{C}^*)^n$ and $w = (w_1, .., w_n) \in \mathbb{Z}^n$, put $z^w = z_1^{w_1}..z_n^{w_n}$. Put $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ and $\mathbb{R}_+^* = \{x \in \mathbb{R} | x > 0\}$. Let $\Delta \subset \mathbb{R}^n$ be an integer convex polytope and $N = Card(\Delta \cap \mathbb{Z}^n) - 1$. Denote by $w_0, ..., w_N$ the integer points of Δ.

Definition 2.2.3. *The toric variety associated to Δ, denoted $Tor(\Delta)$, is the Zarisky closure of the set*

$$\{[z^{w_0} : .. : z^{w_N}] \quad | \quad z \in (\mathbb{C}^*)^n\} \subset \mathbb{C}P^N.$$

Remark 2.2.4. *The most standard definition of a toric variety associated to a polytope is in general different, see for example [Ful93]. Both definitions coincide if the integral points in the polytope Δ generate the lattice, which is the intersection of $\mathbb{Z}_n$ with the minimal affine space containing Δ.*

The action of the torus $(\mathbb{C}^*)^n$ on $Tor(\Delta)$ is given by the formula

$$z \cdot [y_0 : .. : y_N] = [z^{w_0} y_0 : .. : z^{w_N} y_N],$$

and $Tor(\Delta)$ is the closure of the orbit of the point $[1 : .. : 1]$ under this action. The dimension of $Tor(\Delta)$ is equal to the dimension of the polytope Δ.

Remark 2.2.5. *Let Γ be a face of Δ, and let $w_{i_0}, .., w_{i_s}$ be the integer points of Γ. Consider the following embedding of $\mathbb{C}P^s$ into $\mathbb{C}P^N$:*

$$\Phi_{N,s} : \mathbb{C}P^s \longrightarrow \mathbb{C}P^N$$

$$[y_0 : .. : y_s] \mapsto [0 : .. : y_0 : .. : y_s : .. : 0],$$

where, on the right, the variable y_j is in position i_j. The map $\Phi_{N,s}$ gives rise to an embedding of $Tor(\Gamma)$ into $Tor(\Delta)$. In particular, any vertex of Δ gives a point in $Tor(\Delta)$.

Definition 2.2.6. *The standard real structure $conj_\Delta$ on $Tor(\Delta)$ is the restriction to $Tor(\Delta)$ of the standard complex conjugation $conj$ on $\mathbb{C}P^N$, where*

$$conj : \mathbb{C}P^N \longrightarrow \mathbb{C}P^N$$

$$[y_0 : .. : y_N] \mapsto [\overline{y_0} : .. : \overline{y_N}].$$

Example 2.2.7. *Consider the convex hull Δ of the points $(0,0)$, $(0,1)$, $(1,0)$ and $(2,0)$ in $\mathbb{R}^2$. Then $Tor(\Delta)$ is the quadratic cone in $(\mathbb{C}P^3, conj)$.*

Definition 2.2.8. *Let $f = \sum a_i z^i$ be a polynomial in $\mathbb{C}[z_1, .., z_n]$. The convex hull of the set $\{i \in \mathbb{Z}^n | a_i \neq 0\}$ is called the Newton polytope of f. For an integer convex polytope Δ, denote by $\mathcal{P}(\Delta)$ the space of polynomials with Newton polytope Δ.*

Let Δ be an integer convex polytope of dimension n in $(\mathbb{R}_+)^n$ and let $w_0, .., w_N$ be the integer points of Δ. Let $f = \sum\limits_{i=0,..,N} a_i z^{w_i} \in \mathcal{P}(\Delta)$. The polynomial f defines an algebraic hypersurfaces in $Tor(\Delta)$. This hypersurface is a compactification of $Z(f) = \{x \in (\mathbb{C}^*)^n | f(x) = 0\}$.

Definition 2.2.9. *Let $f = \sum a_i x^i \in \mathcal{P}(\Delta)$. Let $\Gamma \subset \mathbb{Z}^n$ be a subset of Δ. The truncation of f to Γ is the polynomial f^Γ defined by $f^\Gamma = \sum\limits_{i \in \Gamma} a_i x^i$.*

Definition 2.2.10. *A polynomial $f \in \mathcal{P}(\Delta)$ is called completely non-degenerate if for any face Γ of Δ (including Δ itself), the hypersurface $Z(f^\Gamma) = \{x \in (\mathbb{C}^*)^n | f^\Gamma(x) = 0\}$ is non-singular.*

Let $f \in \mathcal{P}(\Delta)$, where $\Delta \subset (\mathbb{R}_+)^n$ is of dimension n. Consider the compactification $\overline{Z(f)}$ of $Z(f)$ in $Tor(\Delta)$. If the coefficients of f are real, then $\overline{Z(f)}$ is a real algebraic variety in $(Tor(\Delta), conj_\Delta)$. If f is completely non-degenerate, then for any face Γ of Δ, the set $\overline{Z(f)}$ is transverse to $Tor(\Gamma)$.

Definition 2.2.11. *A subdivision of an integer convex polytope Δ is a set of integer convex polytopes $(\Delta_i)_{i \in I}$ such that:*

- $\bigcup_{i\in I}\Delta_i=\Delta$,
- *if $i,j\in I$, then the intersection $\Delta_i\cap\Delta_j$ is a proper common face of the polytope Δ_i and the polytope Δ_j, or empty.*

Definition 2.2.12. *A subdivision $(\Delta_i)_{i\in I}$ of an integer convex polytope is said to be convex if there exists a convex piecewise-linear function $\nu:\Delta\to\mathbb{R}$ whose domains of linearity concide with the polytopes Δ_i.*

For $\varepsilon=(\varepsilon_1,..,\varepsilon_n)\in(\mathbb{Z}/2\mathbb{Z})^n$, let $s_\varepsilon:\mathbb{R}^n\to\mathbb{R}^n$ be the symmetry defined by

$$s_\varepsilon(v_1,..,v_n)=\big((-1)^{\varepsilon_1}v_1,..,(-1)^{\varepsilon_n}v_n\big).$$

Given an integer convex polytope $\Delta\subset(\mathbb{R}_+)^n$, denote with Δ_* the union

$$\bigcup_{\varepsilon\in(\mathbb{Z}/2\mathbb{Z})^n} s_\varepsilon(\Delta).$$

Suppose $(\Delta_i)_{i\in I}$ to be a convex subdivision of Δ; then it extends to Δ_*.

Notation 2.2.13. *If Γ is a face of Δ_*, then for all integer vectors α orthogonal to Γ and for all $x\in\Gamma$, identify x with $s_\alpha(x)$. Denote by $\hat{\Delta}$ the quotient of Δ_* under these identifications.*

Proposition 2.2.14. *The real part $\mathbb{R}Tor(\Delta)$ of $Tor(\Delta)$ is homeomorphic to $\hat{\Delta}$.*

2.2.2 Viro's patchworking theorem

In this section we define charts of polynomials and state the Viro's patchworking Theorem. The interested reader can find more about Viro's patchworking Theorem in [Vir84a], [Vir84b], [Vir89], [Vir06] and [IS03]. In the general Viro's construction, one starts with a convex subdivision $(\Delta_i)_{i\in I}$ of a convex polytope Δ in $(\mathbb{R}_+)^n$ and an appropriate collection of polynomials f_i, $i\in I$. Then the patchworking method produces an algebraic hypersurface X in $Tor(\Delta)$. The Newton polytope of X is Δ and the topology of the complex part of X (and the real part of X if the polynomials f_i's are real) is described in terms of the topology of the zero point sets of the f_i's.
Let $\Delta\subset(\mathbb{R}_+)^n$ be an integer convex polytope with interior $I(\Delta)$ and vertices $V(\Delta)$, one can define the map $\mu_\Delta:(\mathbb{R}_+^*)^n\to I(\Delta)$, called *moment map*, as follows:

$$\mu_\Delta(z)=\frac{\sum\limits_{i\in\mathbb{Z}^n\cap\Delta} z^i\cdot i}{\sum\limits_{i\in\mathbb{Z}^n\cap\Delta} z^i}.$$

If $dim(\Delta)=n$, then μ_Δ is a diffeomorphism.
Let us consider the diffeomorphism $\zeta:(\mathbb{C}^*)^n\to(\mathbb{R}_+^*)^n\times(S^1)^n$ sending z to $\big((|z_1|,..,|z_n|),(\frac{z_1}{|z_1|},..,\frac{z_n}{|z_n|})\big)$. The inverse of ζ extends to a surjection $\theta:(\mathbb{R}_+^*)^n\times(S^1)^n\to\mathbb{C}^n$. Given any subset E of $(\mathbb{R}^*)^n$, we denote by $\mathbb{C}E$ the subset $\theta(E\times(S_1)^n)$ of $\mathbb{C}^n$.

Definition 2.2.15. *The set $\mathbb{C}\Delta$ is called complexification of Δ.*
The map $M_\Delta=\theta\circ(\mu_\Delta,Id)\circ\zeta$ is called complexification of the moment map.

$$M_\Delta:(\mathbb{C}^*)^n\to(\mathbb{R}_+^*)^n\times(S^1)^n\to I(\Delta)\times(S^1)^n\to\mathbb{C}I(\Delta).$$

Proposition 2.2.16. *The real part $\mathbb{R}\Delta$ of $\mathbb{C}\Delta$ is homeomorphic to Δ_*.*

Proposition 2.2.17. *The map M_Δ is surjective and commutes with the complex conjugation. It is a diffeomorphism when the dimension of Δ is n. The real part of $\mathbb{C}I(\Delta)$ is the image of $(\mathbb{R}^*)^n$ via M_Δ.*

Definition 2.2.18. *Let f be a real polynomial in $\mathcal{P}(\Delta)$ and let $Z(f)$ be the set $\{z \in (\mathbb{C}^*)^n | f(z) = 0\}$. Let us consider the closure of the set $M_\Delta(Z(f)) \subset \mathbb{C}\Delta$; then, its intersection with $\mathbb{R}\Delta$ is called the chart of f and it is denoted with $Ch(f)$.*

Proposition 2.2.19. *For a face Γ of Δ, one has $Ch(f) \cap \mathbb{R}\Gamma = Ch(f^\Gamma)$.*

Define a map $\pi_\Delta : \mathbb{C}\Delta \to Tor(\Delta)$ as follows: given a face $\Gamma \subset \Delta$, or Δ itself, and given $\omega \in \mathbb{C}I(\Gamma)$ such that $\omega = M_\Gamma(z)$, then $\pi_\Delta(\omega)$ is equal to z^i for $i \in \Gamma \cap \mathbb{Z}^n$ and equal to 0 otherwise.

Proposition 2.2.20. *The map π_Δ is continuous, surjective and commutes with the complex conjugation. Furthermore, if the dimension of Δ is n, then ${\pi_\Delta}_{|_{\mathbb{C}I(\Delta)}}$ is a diffeomorphism on its image.*

Let Δ be an integer convex n-dimensional polytope in $(\mathbb{R}_+)^n$ and let $\bigcup_{i\in I} \Delta_i$ a subdivision of Δ. For any $i \in I$, pick a polynomial f_i such that the following properties hold:

- for each $i \in I$ $f_i \in \mathcal{P}(\Delta_i)$ and f_i is completely non-degenerate;
- if $\Gamma = \Delta_i \cap \Delta_j$, then $f_i^\Gamma = f_j^\Gamma$.

Define the polynomial $f = \sum\limits_{w\in\mathbb{Z}^n\cap\Delta} a_w x^w$, such that $f^{\Delta_i} = f_i$ for all $i \in I$.

Theorem 2.2.21 (Viro's patchworking Theorem)**.** *Assume that the subdivision $\bigcup_{i\in I} \Delta_i$ of Δ is convex via a piecewise-linear function $\nu : \Delta \to \mathbb{R}$. Consider the one-parameter family of polynomials, called Viro polynomial,*

$$f_t = \sum_{w\in\mathbb{Z}^n\cap\Delta} a_w t^{\nu(w)} x^w.$$

Then there exists $t_0 > 0$ such that, if $0 < t < t_0$, then f_t is completely non-degenerate and the pairs $\big((\mathbb{R}Tor(\Delta), \pi_\Delta(Ch(f_t))\big)$ and $\big(\mathbb{R}Tor(\Delta), \pi_\Delta(\cup_{i\in I} Ch(f_i))\big)$ are homeomorphic.

We say that f_t (resp. the chart of f_t) is obtained by patchworking $f_1, ..., f_n$ (resp. the charts of $f_1, .., f_n$).

2.2.3 Singular curves

The classification of non-singular real algebraic curves in toric varieties requires the realization of given isotopy types. An effective way of construction of non-singular real algebraic curves with prescribed topology is the perturbation of real algebraic curves with isolated singularities. Since Brusotti's work ([Bru21]), it is known that if a curve has only non-degenerate double points, one can perturb any of them independently from the others. Two effective construction methods, which rely on Brusotti's theorem, are those of Harnack and Hilbert (see respectively [Har76] and [Hil33]); in fact, they gave an algorithmic way to construct real algebraic maximal curves of any degree arranged in $\mathbb{R}P^2$ with respect to a real line, resp. to a non-singular real conic.

The Viro's method has provided a way to perturb more complicated singularities, called *Newton non-degenerate* ([IAG+98]). For example, perturbing real plane curves of degree 7 with two real singular points of type J_{10}, Viro achieved the classification of real schemes realizable by non-singular real algebraic curves of degree 7 ([Vir84a]) and the realization of some real scheme realized by non-singular real algebraic curves of degree 8 in $\mathbb{R}P^2$ ([Vir89]). In Proposition 3.4.10, we exploit a perturbation of the real 5-fold singularity ([Vir86]).

Moreover, in [Shu98], [Shu05] and [Shu06], Shustin proved that Theorem 2.2.21, under some conditions on the singularities, remains true even if the f_i and their truncations have *generalized Newton non-degenerate* singularities in $(\mathbb{C}^*)^n$. In this way, one can construct via the patchworking method singular and non-singular algebraic hypersurfaces in toric varieties.

2.2.3.1 Patchworking charts with a tangency point

The interested reader can find the results of this section in all their generalities in [Shu05]. Here, we only present an application of [Shu05, Theorem. 5] in a particular case.

Let Δ be a convex polygon in $\mathbb{R}^2_+$ and let $Tor(\Delta)$ be its associated toric variety. Let $S : \Delta = \bigcup_{i=1}^k \Delta_i$ be a convex subdivision of Δ. Let

$$f_i = \sum_{(j,h)\in\Delta_i\cap\mathbb{Z}^2} a_{j,h}x^j y^h$$

be real polynomials, with $a_{j,h} \in \mathbb{R}$ and such that $C_i = \{f_i = 0\} \subset Tor(\Delta_i)$ are non-singular real algebraic curves.

Suppose that there exist some faces $\Gamma_{st} \subset \Delta_s \cap \Delta_t$ such that $\Gamma_{st} \not\subset \partial\Delta$ and $z_{st} \in Tor(\Gamma_{st}) \cap C_j$ is a real tangency point of C_j with $Tor(\Gamma_{st})$, for $j = s, t$, and locally as depicted in Fig. 2.2 on the right. Furthermore, suppose that, out of the tangency points z_{ts}, each curve C_i crosses $Tor(\Gamma')$ transversely for any face $\Gamma' \subset \Delta_i$, with $i = 1, .., k$. Gluing the charts of the C_i's, one does not obtain a chart of a polynomial. Let us consider the following topological construction: replace a neighborhood of the tangency points and of their symmetric points in Δ_* (Fig. 2.2) with a *deformation pattern*, i.e. two disks as depicted in *a*) or *b*) of Fig. 2.3. Then, Shustin ([Shu05]) proved that such topological construction is realizable algebraically and it produces a chart of a non-singular real polynomial in $Tor(\Delta)$.

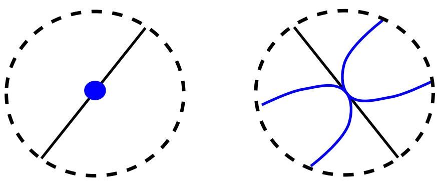


Figure 2.2:

2.2.4 Patchworking of surfaces

We can not directly apply Theorem 2.2.21 to construct real algebraic curves in toric surfaces with a real non-toric structure or in real non-toric surfaces. Then, as a possible approach of construction, one can try to degenerate the ambient surface to a reducible surface on which one can use patchworking method. In

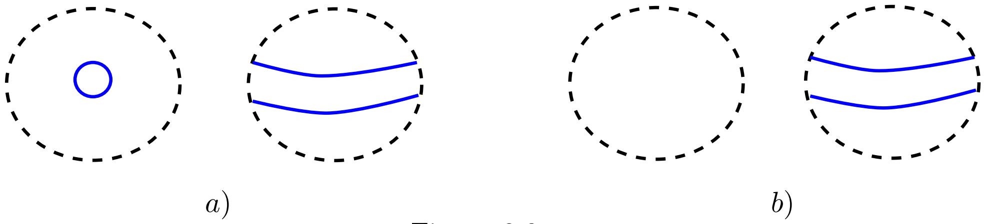


Figure 2.3:

[ST06a] and [ST06b], Shustin and Tyomkin proved a variant of patchworking method which allows, under some conditions, to construct real algebraic curves on real algebraic surfaces, where such surfaces are not required to be real toric surfaces. An application of the results of Shustin and Tyomkin can be found in [BDIM18], where the authors gave constructions of real algebraic curves whose real part consists of a finite number of real non-degenerate double points in $\mathbb{C}P^2$ and in the quadric ellipsoid. We present a particular version of [ST06a, Theorem 2.8], which we exploit in Theorem 5.4.6.

Theorem 2.2.22 (Weak patchworking Theorem.). *We are given the following data:*

- *a one-parameter real flat family of projective surfaces $\pi : X \to T$ over a smooth base T;*
- *a family of invertible sheaves $\mathcal{L}_t$ on $X_t = \pi^{-1}(t)$, i.e. an invertible sheaf $\mathcal{L}$ on X, and*
- *a section $\xi_0 \in H^0(X_0, \mathcal{L}_0)$.*

Assume that our data satisfy the following properties

(1) *X_t is real reduced and irreducible for any $t \neq 0$;*

(2) *$X_0 = \Lambda^1 \cup \Lambda^2$ is a union of two real reduced and irreducible surfaces.*

We denote the zero set of ξ_0 by C_0. Then $C_0 = C_0^1 \cup C_0^2$, where $C_0^i = C_0 \cap \Lambda^i$, with $i = 1, 2$.

(i) *C_0 is a real algebraic curve.*

(ii) *C_0^i is a non-singular real reduced curve in Λ^i, with $i = 1, 2$.*

(iii) *$C_0 \cap \Lambda^1 \cap \Lambda^2$ is reduced.*

(iv) *For any $p \in C_0 \cap \Lambda^1 \cap \Lambda^2$, there exists an open analytic neighborhood $p \in U \subset X$ such that $X_0 \cap U \subset U$ is a normal crossing divisor.*

Assume that

$$H^1(X_0, \mathcal{L}_0) = 0.$$

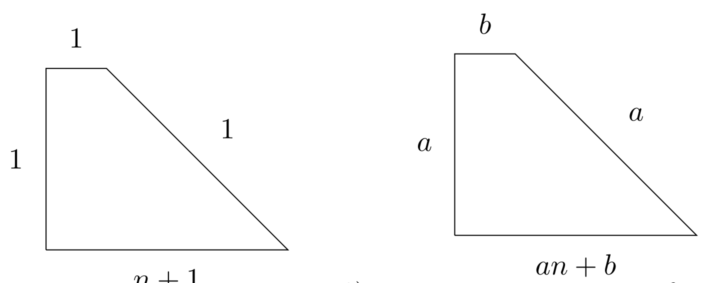
Then there exists some open neighborhood $U_\varepsilon(0) \subset T$ and a real flat family of non-singular real algebraic curves $C_t \in |\mathcal{L}_t|$, with $t \in U_\varepsilon(0)$.

2.3 Hirzebruch surfaces

A Hirzebruch surface is a compact complex surface which admits a holomorphic fibration over $\mathbb{C}P^1$ with fiber $\mathbb{C}P^1$ ([Bea83]) Every Hirzebruch surface is biholomorphic to exactly one of the surfaces $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathbb{C})$ for $n \geq 0$. The surface Σ_n admits a natural fibration

$$\pi_n : \Sigma_n \to \mathbb{C}P^1$$

with fiber $\mathbb{C}P^1 =: F_n$. Denote by B_n, resp. E_n, the section $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \{0\})$, resp. $\mathbb{P}(\{0\} \oplus \mathbb{C})$. The self-intersection of B_n (resp. E_n and F_n) is n (resp. $-n$ and 0). When $n \geq 1$, the exceptional divisor E_n determines uniquely the Hirzebruch surface since it is the only irreducible and reduced algebraic curve in Σ_n with negative self-intersection.
For example $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$. The Hirzebruch surface Σ_1 is the complex projective plane blown-up at a point, and Σ_2 is the quadratic cone with equation $Q_0 : X^2 + Y^2 - Z^2 = 0$ blown-up at the node in $\mathbb{C}P^3$. The fibration of Σ_2 (resp. of Σ_1) is the extension of the projection from the blown-up point to a hyperplane section (resp. to a line) which does not pass through the blown-up point.
The group $H_2(\Sigma_n; \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and is generated by the classes $[B_n]$ and $[F_n]$. An algebraic curve C in Σ_n is said to be of bidegree (a, b) if it realizes the homology class $a[B_n] + b[F_n]$ in $H_2(\Sigma_n; \mathbb{Z})$. Note that $[E_n] = [B_n] - n[F_n]$ in $H_2(\Sigma_n; \mathbb{Z})$. An algebraic curve of bidegree $(3, 0)$ on Σ_n is called a *trigonal curve*.
We can obtain Σ_{n+1} from Σ_n via a birational transformation $\beta_n^p : \Sigma_n - - \to \Sigma_{n+1}$ which is the composition of a blow-up at a point $p \in E_n \subset \Sigma_n$ and a blow-down of the strict transform of the fiber $\pi_n^{-1}(\pi_n(p))$.
The surface Σ_n is also the projective toric surface which corresponds to the polygon with vertices $(0, 0), (0, 1), (1, 1), (n+1, 0)$, depicted in Fig. 2.4 a) where the number labeling an edge corresponds to its integer length. The Newton polygon of an algebraic curve C of bidegree (a, b) on Σ_n, lies inside the trapeze with vertices $(0, 0), (0, a), (b, a), (an + b, 0)$ as in Fig. 2.4 b). The surface Σ_n is canonically endowed by a real structure induced by the standard complex conjugation in $(\mathbb{C}^*)^2$. For this real structure the real part of Σ_n, denoted by $\mathbb{R}\Sigma_n$, is a torus if n is even and a Klein bottle if n is odd. We will depict $\mathbb{R}\Sigma_n$ as a quadrangle whose opposite sides are identified in a suitable way and the horizontal sides will represent $\mathbb{R}E_n$. Moreover, let C be any type I real algebraic curve in Σ_n; the depicted orientation on $\mathbb{R}C$ will denote a complex orientation.
The restriction of π_n to $\mathbb{R}\Sigma_n$ defines a S^1-bundle over S^1 that we denote by



a) Polygon defining Σ_n b) The Newton polygon of a curve of bidegree (a, b) on Σ_n

Figure 2.4:

$\mathcal{L}$. We are interested in the isotopy types with respect to $\mathcal{L}$ of real algebraic curves in $\mathbb{R}\Sigma_n$.

Definition 2.3.1. • *Two arrangements of circles and points immersed in $\mathbb{R}\Sigma_n$ are $\mathcal{L}$-isotopic if there exists an isotopy of $\mathbb{R}\Sigma_n$ which brings one arrangement to the other, each line of $\mathcal{L}$ to another line of $\mathcal{L}$ and whose restriction to $\mathbb{R}E_n$ is an isotopy of $\mathbb{R}E_n$.*

- *An arrangement of circles and points immersed in $\mathbb{R}\Sigma_n$ up to $\mathcal{L}$-isotopy of $\mathbb{R}\Sigma_n$ is called an $\mathcal{L}$-scheme.*

- *A $\mathcal{L}$-scheme is realizable by a real algebraic curve of bidegree (a, b) in Σ_n if there exists such a curve whose real part is $\mathcal{L}$-isotopic to the arrangement of circles and points in $\mathbb{R}\Sigma_n$.*

- *A trigonal $\mathcal{L}$-scheme is a $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ which intersects each fiber in 1 or 3 real points counted with multiplicities and which does not intersect $\mathbb{R}E_n$.*

- *A trigonal $\mathcal{L}$-scheme η in $\mathbb{R}\Sigma_n$ is hyperbolic if it intersects each fiber in 3 real points counted with multiplicities.*

2.4 Dessins d'enfant

Orevkov in [Ore03] has formulated the existence of real algebraic trigonal curves realizing a given trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ in terms of the existence of a real rational graph on $\mathbb{C}P^1$. Later on, Degtyarev, Itenberg and Zvonilov in [DIZ14] have given a general way to determine if such real algebraic trigonal curves are of type I or II.
In Section 3.4.1, we will exploit such construction techniques in order to construct real algebraic trigonal curves in rational geometrically ruled surfaces. Therefore, we present here some results of [Ore03] and [DIZ14].

Definition 2.4.1. *Let n be a fixed positive integer. We say that a graph Γ is a real trigonal graph of degree n if*

- *it is a finite oriented connected graph embedded in $\mathbb{C}P^1$, invariant under the standard complex conjugation of $\mathbb{C}P^1$;*

- *it is decorated with the additional following structure:*

  - ▶ *every edge of Γ is colored solid, bold or dotted;*
  - ▶ *every vertex of Γ is $\bullet$, $\circ$, $\times$ (said essential vertices) or monochrome*

  and satisfying the following conditions:

  (1) any vertex is incident to an even number of edges; moreover, any $\circ$-vertex (resp. $\bullet$-vertex) to a multiple of 4 (resp. 6) number of edges;

  (2) for each type of essential vertices, the total sum of edges incident to the vertices of a same type is $12n$;

  (3) the orientations of the edges of Γ form an orientation of $\partial(\mathbb{C}P^1 \setminus \Gamma)$ which is compatible with an orientation of $\mathbb{C}P^1 \setminus \Gamma$ (see Fig. 2.6);

  (4) all edges incidents to a monochrome vertex have the same color;

  (5) $\times$-vertices are incident to incoming solid edges and outgoing dotted edges;

  (6) $\circ$-vertices are incident to incoming dotted edges and outgoing bold edges;

(7) $\bullet$-vertices are incident to incoming bold edges and outgoing solid edges.

Given a trigonal $\mathcal{L}$-scheme η in $\mathbb{R}\Sigma_n$, we are interested in constructing a real trigonal curve C in Σ_n such that $\mathbb{R}C$ and η are $\mathcal{L}$-isotopic in $\mathbb{R}\Sigma_n$.

Definition 2.4.2. *Let η be a trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$. Let us consider the restriction of the projection π_n (see Section 2.3) to $\mathbb{R}\Sigma_n$. Thanks to $\pi_{n|_{\mathbb{R}\Sigma_n}}$ we can encode η by a colored oriented graph $\overline{\Gamma}$ on $\mathbb{R}P^1 \subset \mathbb{C}P^1$ in the following way (in Fig. 2.5 the dashed lines denote fibers of $\pi_{n|_{\mathbb{R}\Sigma_n}}$):*

(1) *To each fiber of $\pi_{n|_{\mathbb{R}\Sigma_n}}$ intersecting η in two points we associate a $\times$-vertex on $\mathbb{R}P^1$.*

(2) *Let F_1, F_2 be two fibers of $\pi_{n|_{\mathbb{R}\Sigma_n}}$ intersecting η in two points such that η, up to $\mathcal{L}$-isotopy, is locally as depicted in Fig. 2.5 b) or c). Let F_3 be another fiber between F_1, F_2. Then, we associate to F_3 a $\circ$-vertex on $\mathbb{R}P^1$. Moreover, if between F_1 and F_2 each other fiber intersects η in only one real point (as in b) of Fig. 2.5), then we associate to a fiber between F_1 and F_3 (resp. F_3 and F_2) a $\bullet$-vertex on $\mathbb{R}P^1$. Between $\bullet$ and $\circ$-vertices we put bold edges.*

(3) *Except for the fibers of $\pi_{n|_{\mathbb{R}\Sigma_n}}$ to which we associate essential vertices and bold edges, to a fiber which intersects η in three distinct real points (resp. only one real point) we associate dotted (resp. solid) edges on $\mathbb{R}P^1$.*

(4) *The orientations of the edges incident to a vertex are in an alternating order. In particular, the orientations of the edges incident to an essential vertex are respectively as described in* (5), (6), (7) *of Definition 2.4.1.*

The graph $\overline{\Gamma}$, called real graph, is considered up to isotopy of $\mathbb{R}P^1$, namely it is determined by the order of its colored vertices since the edges are determined by the color of their adjacent vertices.
We say that $\overline{\Gamma}$ is completable in degree n *if there exists a complete real trigonal graph Γ of degree n such that $\Gamma \cap \mathbb{R}P^1 = \overline{\Gamma}$.*

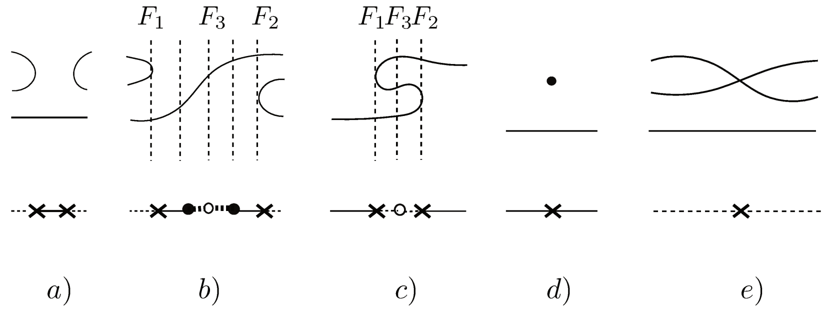


Figure 2.5: Local topology of trigonal $\mathcal{L}$-schemes and their corresponding real graphs.

Theorem 2.4.3 ([Ore03]). *A trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_n$ is realizable by a real algebraic trigonal curve if and only if its real graph is completable in degree n.*

Given a real graph $\overline{\Gamma}$, we depict only the completion to a real trigonal graph Γ on a hemisphere of $\mathbb{C}P^1$ since Γ is symmetric with respect to the standard complex conjugation. Moreover, we can omit orientations in figures representing real trigonal graphs because each vertex is adjacent to an even number of edges oriented in an alternating order as, for example, depicted in Fig. 2.6 and such orientations are compatible with each others.

Theorem 2.4.3 is improved in [DIZ14] in order to check if a given trigonal

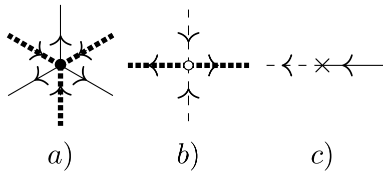


Figure 2.6: Colored vertices of a real trigonal graph.

$\mathcal{L}$-scheme is realizable by a real trigonal curve of type I. We say that a real algebraic singular curve is of type I (resp. of type II) if its normalization is of type I (resp. of type II).

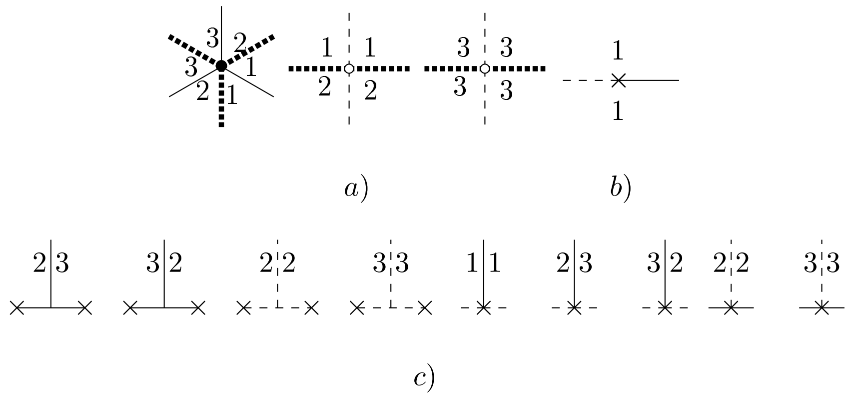


Figure 2.7: Type I labeling.

Definition 2.4.4. *Let Γ be a real trigonal graph of degree n. We say that Γ is of type I if we can label each connected component of $\mathbb{C}P^1 \setminus \Gamma$, with the numbers* $1, 2$ *or* 3*, such that:*

- *neighboring connected components of a $\bullet$-vertex, or a $\circ$-vertex of Γ, are labeled as depicted in one of the pictures in Fig. 2.7 a);*
- *neighboring connected components of a $\times$-vertex, which does not belong to $\Gamma \cap \mathbb{R}P^1$, are labeled as depicted in Fig. 2.7 b);*
- *neighboring connected components of $\times$-vertices, belonging to $\Gamma \cap \mathbb{R}P^1$, are labeled as depicted in in Fig. 2.7 c).*

Otherwise, we say that Γ is of type II.

The original statement in [DIZ14] of the following theorem treats only the case of non-singular real trigonal curve, but it is possible to extend it to real nodal trigonal curves.

Theorem 2.4.5 ([DIZ14])**.** *A non-hyperbolic trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_n$ is realizable by a real trigonal curve of type I (resp. of type II) if and only if its real graph has a completion in degree n which is of type I (resp. type II).*

Remark 2.4.6. *A non-hyperbolic trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_n$ is realizable by an irreducible real trigonal curve of type I if and only if there exists at least one completion in degree n of its real graph which has a unique type I labeling (see [DIZ14]). So, later on, each time we have to assign a labeling to a real trigonal graph of type I, we could label only one component.*

Remark 2.4.7. *There exists a completion in degree n of the real graph of a non-hyperbolic trigonal $\mathcal{L}$-scheme η on $\mathbb{R}\Sigma_n$ such that it has at least two type I labelings if and only if there exists a reducible real trigonal curve realizing η (see [Jar18]); furthermore, such trigonal curve has to be the union of a real curve of bidegree $(2,0)$ and a real curve of bidegree $(1,0)$ in Σ_n.*

Remark 2.4.8. *If a hyperbolic trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ is realizable by a real trigonal curve C in Σ_n, then the curve C is of type I because the projection $\pi_n : \Sigma_n \to \mathbb{C}P^1$ (see Section 2.3) gives a totally real morphism on $\mathbb{C}P^1$.*

2.4.0.1 Gluing real trigonal graphs

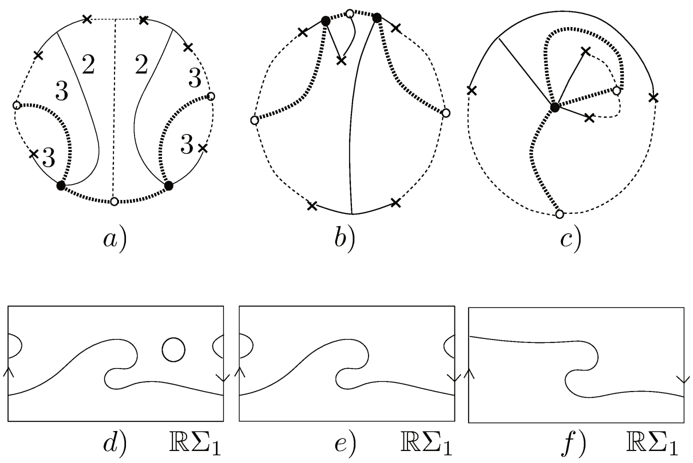


Figure 2.8: Cubic trigonal graphs.

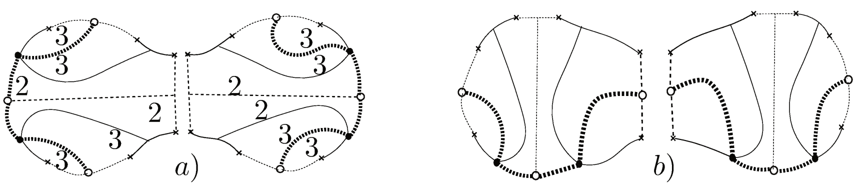


Figure 2.9: How gluing two cubic trigonal graphs.

We call *cubic trigonal graph of type I* (resp. *type II*) a real trigonal graph of degree 1 and type I (resp. type II). The graph in Fig. 2.8 a) is a cubic trigonal graph of type I, it has a unique type I labeling and associated trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_1$ as depicted in Fig. 2.8 d). While the graphs depicted in Fig. 2.8 b) and c) are of type II and have associated trigonal $\mathcal{L}$-schemes on $\mathbb{R}\Sigma_1$ as depicted respectively in Fig. 2.8 e) and f).

Let Γ_1 (resp. Γ_2) be a real trigonal graph. Denote by D_1 (resp. D_2) the disk on which one of the two symmetric halves of Γ_1 (resp. Γ_2) lies. Consider the disjoint union $\Gamma_1 \sqcup \Gamma_2 \subset D_1 \sqcup D_2$. Let $I_i \subset D_i$, $i = 1, 2$, be a segment in $\mathbb{R}P^1$ whose endpoints are not vertices of Γ_i and such that I_i contains a single $\circ$-vertex or a monochrome dashed vertex. Let $\phi : I_1 \to I_2$ be an isomorphism preserving orientation, i.e. $\Gamma_1 \cap I_1 \to \Gamma_2 \cap I_2$ is an isomorphism preserving the types of vertices and edges, and preserving orientation. Consider the quotient $D_1 \sqcup_\phi D_2 = D_1 \sqcup D_2/(x \sim \phi(x))$ and $\Gamma_\phi \subset D_1 \sqcup_\phi D_2$ the quotient of the image of $\Gamma_1 \sqcup \Gamma_2$. We call such operation *gluing*. The gluing of real trigonal graphs is still a real trigonal graph (see [DIK08, Section 5.6] for details).

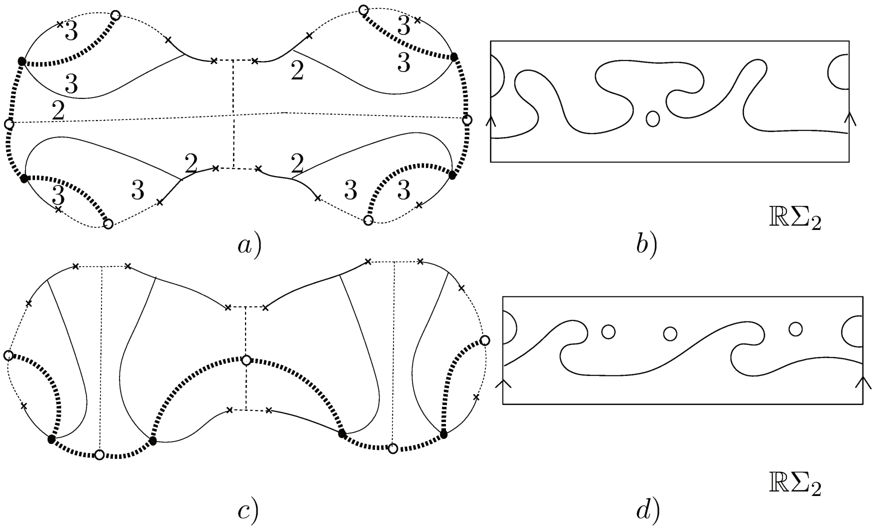


Figure 2.10: Gluing of two cubic trigonal graphs and associated trigonal $\mathcal{L}$-scheme.

One can remark that there is a finite number of real trigonal graphs of degree n that can be obtained as gluing of n cubic trigonal graphs.
Gluing type I real trigonal graphs, which are glued to each other along vertices whose neighboring connected components have the same labels, we get a type I real trigonal graph. As example, look at the gluing of two cubic trigonal graphs of type I in Fig. 2.9 a), resp. b), and 2.10 a), resp. c). The obtained graphs are real trigonal graphs of degree 2 and type I. The respective associated trigonal $\mathcal{L}$-schemes are depicted in Fig. 2.10 b) and d).

Chapter 3

Real algebraic curves of bidegree (5,5) on the quadric ellipsoid

3.1 Introduction

Let X be $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the anti-holomorphic involution

$$\begin{array}{cccc} \sigma : & X & \longrightarrow & X \\ & (x,y) & \longmapsto & (\overline{y},\overline{x}) \end{array}$$

where $x = [x_0 : x_1]$ and $y = [y_0 : y_1]$ are in $\mathbb{C}P^1$ and $\overline{x} = [\overline{x_0} : \overline{x_1}]$ and $\overline{y} = [\overline{y_0} : \overline{y_1}]$ are respectively the images of x and y via the standard complex conjugation on $\mathbb{C}P^1$. The real part of X is homeomorphic to S^2. It is well known that X is isomorphic to the quadric ellipsoid in $\mathbb{C}P^3$. A non-singular real algebraic curve A on X is defined by a bi-homogeneous polynomial of bidegree (d,d)

$$P(x_0,x_1,y_0,y_1) = \sum_{i,j=1}^{d,d} a_{i,j} x_1^i x_0^{d-i} y_1^j y_0^{d-j}$$

where d is a positive integer and the coefficients satisfy $a_{i,j} = \overline{a_{j,i}}$.

The connected components of $\mathbb{R}A$ are called *ovals*. We are interested in the classification of the oval arrangements of non-singular real algebraic curves in X. Due to the Harnack-Klein's inequality and the adjunction formula, the number of the connected components of $\mathbb{R}A$ is bounded by $(d-1)^2+1$.

Definition 3.1.1. *Let A be a non-singular real algebraic curve of bidegree (d,d) on X. We say that A has real scheme $\mathcal{S}$ if the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes $\mathcal{S}$.*

Real algebraic non-singular curves of bidegree $(4,4)$ (and less) in X have been classified in [GS80]. In this chapter, we give the classification, up to homeomorphism, of the topological types of the pair $(\mathbb{R}X, \mathbb{R}A)$ where A is a non-singular real algebraic curve of bidegree $(5,5)$ in X. If A is a maximal curve, we have the following result.

Theorem 3.1.2 (Maximal curves)**.** *Let A be a non-singular real algebraic M-curve of bidegree $(5,5)$ on X. Then the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the following real schemes:*

$$\alpha \ \sqcup \ \langle \beta \rangle \ \sqcup \ \langle \gamma \rangle, \ \alpha \equiv 1 \pmod 4, \ \textit{with } \alpha + \beta + \gamma = 15.$$

Moreover, all such real schemes are realizable by non-singular real algebraic M-curves of bidegree $(5,5)$.

The following theorems concerns real algebraic $(M-i)$-curves of bidegree $(5,5)$ in X which are separating or non-separating.

Theorem 3.1.3 ($(M-1)$ curves)**.** *Let A be a non-singular real algebraic $(M-1)$-curve of bidegree $(5,5)$ on X. Then the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the following real schemes:*

$$\alpha \ \sqcup \ \langle\beta\rangle \ \sqcup \ \langle\gamma\rangle, \ \alpha \equiv 0 \text{ or } 1 \pmod 4, \ \textit{with } \alpha+\beta+\gamma = 14.$$

Moreover, all such real schemes are realizable by non-singular real algebraic $(M-1)$-curves of bidegree $(5,5)$.

Theorem 3.1.4 ($(M-2)$-curves)**.** *Let A be a non-singular real algebraic $(M-2)$-curve of bidegree $(5,5)$ on X. If A is of type I, then the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the following real schemes:*

$$(1) \ \alpha \ \sqcup \ \langle\beta\rangle \ \sqcup \ \langle\gamma\rangle, \ \alpha \equiv 0 \pmod 2, \ \textit{with } \alpha+\beta+\gamma = 13;$$

if A is of type II, then one of the following ones:

$$(2) \ \alpha \ \sqcup \langle\beta\rangle \ \sqcup \ \langle\gamma\rangle, \ \alpha \not\equiv 2 \pmod 4, \ \textit{with } \alpha+\beta+\gamma = 13.$$

Moreover, all the real schemes in (1) and (2) are realizable by non-singular real algebraic $(M-2)$-curves of bidegree $(5,5)$ respectively of type I and II.

Theorem 3.1.5 (Type I and II curves)**.** *Let A be a non-singular real algebraic $(M-i)$-curve of bidegree $(5,5)$ on X, where $3 \leq i \leq 17$. If A is of type II, then the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the following real schemes:*

$$(1) \ 0 \textit{ and } 1,$$

$$(2) \ \alpha \ \sqcup \ \langle\beta\rangle \ \sqcup \ \langle\gamma\rangle, \ \textit{with } \alpha+\beta+\gamma = 17-(i-2);$$

if A is of type I, then $i = 4, 6, 8, 10, 12$ and the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the following real schemes:

$$(3) \ \langle\langle\langle\langle 1\rangle\rangle\rangle\rangle,$$

$$(4) \ \alpha \ \sqcup \ \langle\beta\rangle \ \sqcup \ \langle\gamma\rangle, \ \textit{with } \alpha \equiv 0 \pmod 2 \ \textit{when } \alpha+\beta+\gamma = 5, 9,$$

$$\textit{and with } \alpha \equiv 1 \pmod 2 \ \textit{when } \alpha+\beta+\gamma = 7, 11.$$

Moreover, the real schemes in $(1), (2)$ and $(3), (4)$ are realizable by non-singular real algebraic curves of bidegree $(5,5)$ respectively of type II and I.

Remark 3.1.6. *It is easy to see that the classification up to homeomorphism of the pairs $(\mathbb{R}X, \mathbb{R}A)$ in Theorems 3.1.2, 3.1.3, 3.1.4, 3.1.5 is equivalent to the classification up to isotopy.*

Previous results concerning the restrictions part of the classification are given in Proposition 3.2.1, Theorem 3.2.3 and Proposition 3.2.4. Such results give a system of restrictions for real schemes of algebraic curves of bidegree $(5,5)$ in X. The content of this chapter is the proof that such system of restrictions is complete. Therefore, the main part of the paper concerns the construction of separating and non-separating real algebraic curves realizing all real schemes which are not prohibited by restrictions in Section 3.2.

3.2 Restrictions

3.2.1 Restriction on the depth of nests

Proposition 3.2.1. *[DK00, Proposition 4.9.2] Let A be a non-singular real algebraic curve of bidegree (d,d) on X. Then the total number of ovals in any collection of three disjoint nests of $\mathbb{R}A$ does not exceed d.*

Proposition 3.2.1 implies in particular that the maximal depth for a nest of such a curve A is d. Furthermore, it is well known that if A is of type I and has d ovals, it has a nest of maximal depth d (see Corollary 3.2.5).
The Harnack-Klein's inequality combined with Proposition 3.2.1 immediately implies the following corollary.

Corollary 3.2.2. *Let A be a non-singular real algebraic curve of bidigree $(5,5)$ on X. Then the pair $(\mathbb{R}A, \mathbb{R}X)$ realizes one of the following real schemes:*

- 0 *and* 1,
- $\alpha \sqcup \langle\beta\rangle \sqcup \langle\gamma\rangle$, *for* $0 \leq \alpha+\beta+\gamma \leq 15$,
- $\langle\langle\langle\langle 1\rangle\rangle\rangle\rangle$.

3.2.2 Congruences and complex orientations formula on the quadric ellipsoid

Mikhalkin ([Mik94]) provided new restrictions on the topology of real algebraic curves of bidegree (d,d), for all odd integer d, in X, proving the following theorem.

Theorem 3.2.3. *[Mik94, Theorem 1] Let A be a non-singular real algebraic curve of bidegree (d,d) on X, with d odd. Let B be a disjoint union of connected components of $\mathbb{R}X \setminus \mathbb{R}A$ such that $\mathbb{R}A$ bounds B.*

- *If A is a M-curve, then*
$$\chi(B) \equiv \frac{d^2+1}{2} \pmod 8.$$
- *If A is a $(M-1)$-curve, then*
$$\chi(B) \equiv \frac{d^2+1}{2} \pm 1 \pmod 8.$$
- *If A is a $(M-2)$-curve and*
$$\chi(B) \equiv \frac{d^2-7}{2} \pmod 8,$$
then A is of type I.
- *If A is of type I, then*
$$\chi(B) \equiv 1 \pmod 4.$$

Let A be a non-singular real algebraic type I curve of bidegree (d,d) on X. Fix a complex orientation on $\mathbb{R}A$. Since any pair of ovals of $\mathbb{R}A$ bounds an annulus in $\mathbb{R}X$, we distinguish two types of pairs: denote by Π_- (respectively

Π_+) the number of pairs of ovals realizing the same (resp. different) first homology class of the corresponding annulus. Zvonilov in [Zvo83] gave a complex orientations formula for type I non-singular real algebraic curves on X. This formula depends on the choice of an auxiliary point in $\mathbb{R}X \setminus \mathbb{R}A$. Afterwards, Orevkov in [Ore07] reformulated it with no dependence on the choice of an auxiliary point.

Proposition 3.2.4. *[Zvo83], [Ore07, Proposition 1.2] Let A be a non-singular real algebraic type I curve of bidegree (d, d) on X. Denoting by l the number of connected components of $\mathbb{R}A$, one has the following complex orientations formula:*

$$2(\Pi_+ - \Pi_-) = l - d^2 \tag{3.1}$$

Corollary 3.2.5. *[Ore07, Proposition 1.3] Let A be a non-singular real algebraic type I curve of bidegree (d, d) on X. Then $\mathbb{R}A$ has at least d connected components. Furthermore, if $\mathbb{R}A$ has d connected components, it consists of a nest of maximal depth d.*

Corollary 3.2.2 and Theorem 3.2.3 give a complete system of restrictions for real schemes of non-singular real algebraic curves of bidegree $(5, 5)$ on X. Moreover, Theorem 3.2.3 and Proposition 3.2.4 allow us to give even finer restrictions on which real schemes, listed in Corollary 3.2.2, may be realized by type I (resp. type II) non-singular real algebraic curves of bidegree $(5, 5)$ on X. Therefore, given a non-singular real algebraic curve A of bidegree $(5, 5)$ on X, the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes one of the real schemes listed in Theorems 3.1.2, 3.1.3, 3.1.4, 3.1.5.
In the next sections we pass to the construction part of the classification.

3.3 The quadric ellipsoid and the second Hirzebruch surface

We explain how to construct a real algebraic curve of bidegree (d, d) on the quadric ellipsoid in $\mathbb{C}P^3$ starting from a real algebraic curve of bidegree $(d, 0)$ in Σ_2 endowed with the real canonical structure (see Section 2.3).
Let C be a real algebraic curve of degree $(d, 0)$ in Σ_2. Now, cut $\mathbb{R}\Sigma_2$ along $\mathbb{R}E_2$, as depicted in Fig. 3.1 a), and glue two discs D_1, D_2 as depicted in Fig. 3.1, b). By this construction we obtain a 2-sphere S^2. Moreover, from the arrangement of the triplet $(\mathbb{R}\Sigma_2, \mathbb{R}E_2, \mathbb{R}C)$ we obtain an arrangement B of embedded circles in S^2. As example, look at Fig 3.2 where we obtain the arrangement $1 \sqcup \langle 1 \rangle$ in S^2. The following proposition states that such topological construction is realizable algebraically.

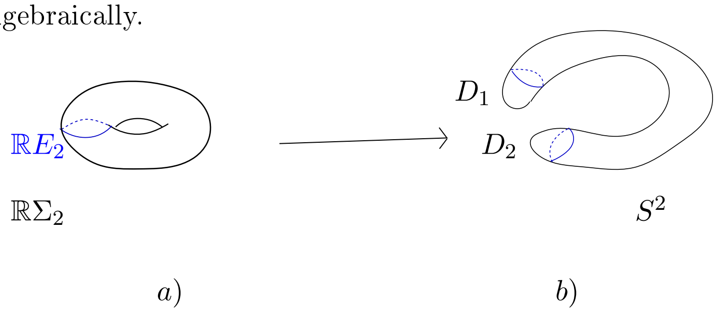


Figure 3.1: From a torus to a 2-sphere

Proposition 3.3.1. *Let C be a non-singular real algebraic curve of bidegree $(d, 0)$ in Σ_2. Let B be the real scheme on the sphere S^2 obtained from the pair*

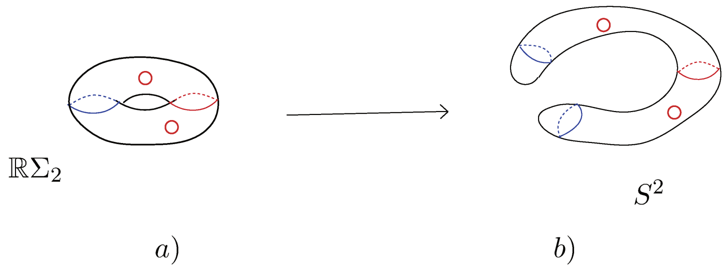


Figure 3.2: Example: from an arrangement of embedded circles in $\mathbb{R}\Sigma_2$ to an arrangement in S^2

$(\mathbb{R}\Sigma_2, \mathbb{R}C)$ by the construction above. Then, B is realizable by a real algebraic curve of bidegree (d, d) on the quadric ellipsoid in $\mathbb{C}P^3$.

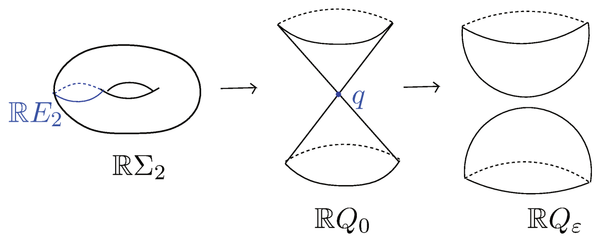


Figure 3.3:

Proof. Let $[X : Y : Z : W]$ be the homogeneous coordinates in the complex projective space. Let Q_0 be the quadratic cone with equation $X^2+Y^2-Z^2 = 0$ in $\mathbb{C}P^3$. Recall that we obtain Σ_2 blowing-up Q_0 at the point $q = [0 : 0 : 0 : 1]$. The image of C via the blow-down is a real algebraic curve $\tilde{C}$ of degree $2d$ in $\mathbb{C}P^3$. Since the dimension of the space of curves of bidegree $(d, 0)$ in Σ_2 is equal to the dimension of the space of complete intersections of the surfaces of degree d in $\mathbb{C}P^3$ with Q_0, the curve $\tilde{C}$ is the intersection of a non-singular real algebraic surface S_d of degree d, not passing through the node of Q_0, and Q_0. Observe that we can perturb Q_0 to the quadric ellipsoid Q_ε of equation $X^2 + Y^2 - Z^2 = -\varepsilon W^2$, where $\varepsilon > 0$; see Fig. 3.3. Since a real algebraic curve of bidegree (d, d) on Q_ε is the intersection of the quadric ellipsoid and a surface of degree d, the intersection of S_d and Q_ε is a real algebraic curve A of bidegree (d, d). Moreover, the pair $(\mathbb{R}Q_\varepsilon, \mathbb{R}A)$ realizes B. □

3.4 Constructions

3.4.1 Trigonal construction

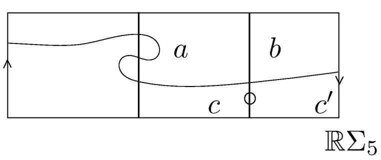


Figure 3.4: The union of a trigonal $\mathcal{L}$-scheme and two fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_5$: $\eta_{a,b,c,c'}$.

In this section we give some intermediate constructions of real algebraic curves that we will need later on.

Proposition 3.4.1. *Let $\eta_{a,b,c,c'}$ be, up to $\mathcal{L}$-isotopy of $\mathbb{R}\Sigma_5$, the union of a trigonal $\mathcal{L}$-scheme with two fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_5$ as depicted in Fig. 3.4, where letters a, b, c, c' denote numbers of ovals. Let h, j and t be non-negative integer numbers. Then, there exist real algebraic trigonal curves in Σ_5 realizing the real schemes $\eta_{a,b,c,c'}$ for all a, b, c, c' such that $0 \leq c + c' \leq h, 0 \leq a \leq j$ and $0 \leq b \leq t$, where h, j and t are the following:*

(1) $j + h + t = 12$ with

- $h = 1$ *and* $j \in \{0, 1, 4, 7, 10, 11\}$,
- $h = 5$ *and* $j \in \{0, 1, 2, 3, 4, 5, 6, 7\}$,
- $h = 9$ *and* $j \in \{0, 1, 2, 3\}$.

(2) $j + h + t = 10$ with

- $h = 0$ *and* $j \in \{4, 6, 8\}$,
- $h = 2$ *and* $j \in \{0, 1, 2, 4, 5, 6, 8\}$,
- $h = 4$ *and* $j \in \{0, 1, 2, 4, 5, 6\}$,
- $h = 6$ *and* $j \in \{0, 1, 2, 4\}$,
- $h = 8$ *and* $j \in \{0, 1, 2\}$.

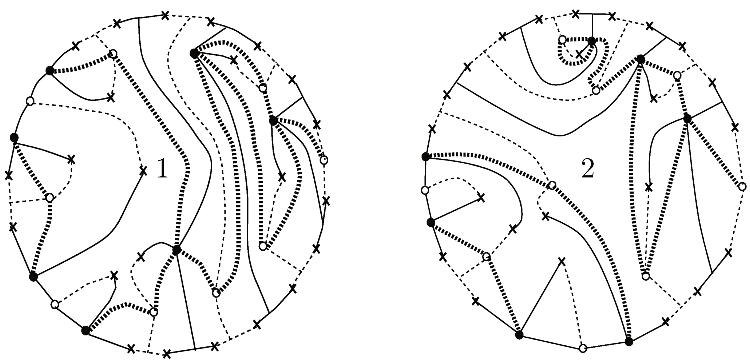
(3) $j + h + t = 8$ with

- $h = 1$ *and* $j = 3$,
- $h = 3$ *and* $j \in \{1, 2, 5\}$.

In particular, such real trigonal curves are of type I for $c + c' = h$, $a = j$ and $b = t$. Moreover, there exist real trigonal curves of type I in Σ_5 realizing $\eta_{a,b,c,c'}$ for $(a, b, c, c') = (1, 5, 0, 0)$ and $(3, 3, 0, 0)$.

Proof. Thanks to Theorems 2.4.3, 2.4.5, if the real graphs associated to $\eta_{a,b,c,c'}$ are completable in degree 5 to a real trigonal graph of type I (resp. II), then there exist real algebraic trigonal curves of type I (resp. type II) realizing $\eta_{a,b,c,c'}$.

We can glue 5 cubic trigonal graphs (see Section 2.4.0.1) in such a way



a) *b*)

Figure 3.5: Real trigonal graphs of degree 5 and type I.

that we obtain type I (resp. type II) real trigonal graphs of degree 5 which complete the real graph associated to $\eta_{a,b,c,c'}$ where a, b, c and c' are such that $c + c' = h$, $a = j$ and $b = t$ (resp. $0 \leq c + c' < h$, $0 \leq a < j$ and $0 \leq b < t$), for h, j and t as listed in $(1) - (3)$ above. Finally, a type I completion (see Remark 2.4.6) of the real graph associated to $\eta_{a,b,c,c'}$ for $(a, b, c, c') = (3, 3, 0, 0)$, resp.$(a, b, c, c') = (1, 5, 0, 0)$, is pictured in a) of Fig. 3.5 a), resp. b). $\square$

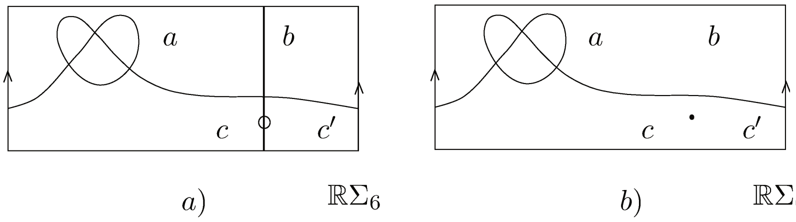


Figure 3.6: $a)$ The union of a trigonal $\mathcal{L}$-scheme and a fiber of $\mathcal{L}$ on $\mathbb{R}\Sigma_6$. $b)$ A nodal trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_6$: $\tilde{\eta}_{a,b,c,c'}$.

Proposition 3.4.2. *Let $\tilde{\eta}_{a,b,c,c'}$ be, up to $\mathcal{L}$-isotopy of $\mathbb{R}\Sigma_6$, a trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_6$ as depicted in b) of Fig. 3.6, where letters a, b, c, c' denote numbers of ovals. Let h, j and t be non-negative integer numbers. Then, there exist real algebraic trigonal curves in Σ_6 realizing the real schemes $\tilde{\eta}_{a,b,c,c'}$ for all a, b, c, c' such that $0 \leq c + c' \leq h, 0 \leq a \leq j$ and $0 \leq b \leq t$, where h, j and t are as listed in Proposition 3.4.1.*

Proof. Thanks to Theorems 2.4.3, 2.4.5, if the real graphs associated to $\tilde{\eta}_{a,b,c,c'}$ are completable in degree 6 to a real trigonal graph of type I (resp. II), then there exist real algebraic trigonal curves of type I (resp. type II) realizing $\tilde{\eta}_{a,b,c,c'}$.
For a, b, c, c' as listed in Proposition 3.4.1, the existence of real trigonal graphs of degree 6 and type I (resp. type II) completing the real graph associated to $\tilde{\eta}_{a,b,c,c'}$, is equivalent to the existence of those of type I (resp. type II) associated to the $\mathcal{L}$-scheme depicted in $a)$ of Fig. 3.6, see [Ore03].
Let ξ be the cubic trigonal graph of type I pictured in $a)$ of Fig. 3.7. Take

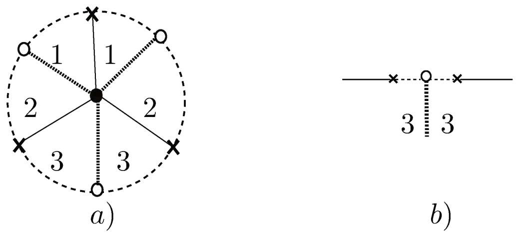


Figure 3.7: $a)$ Real trigonal graph of degree 1 and type I: ξ . $b)$ Local type I labeling.

any real trigonal graph Γ of degree 5 constructed in the proof of Proposition 3.4.1, realizing a trigonal $\mathcal{L}$-scheme $\eta_{a,b,c,c'}$. In a neighborhood of $\Gamma \cap \mathbb{R}P^1$, let us denote by δ the sub-graph of Γ which is as depicted in Fig. 3.7 $b)$ and whose associated $\mathcal{L}$-scheme is the part of $\eta_{a,b,c,c'}$ through which passes one fixed fiber of $\mathcal{L}$; see Fig. 3.4. Glue Γ along the $\circ$-vertex of δ to the $\circ$-vertex, with same labeling if Γ is of type I, of the cubic trigonal graph ξ. The gluing is a real trigonal graph of degree 6 which completes the real graph associated to the union of a trigonal $\mathcal{L}$-scheme with one fiber of $\mathcal{L}$ as depicted in Fig. 3.6 $a)$ on $\mathbb{R}\Sigma_6$, for all a, b, c, c' as in Proposition 3.4.1. Besides, the gluing is of type I (resp. of type II) for all a, b, c, c' for which Γ is of type I (resp. type II). $\square$

Proposition 3.4.3. *Let $\eta_{i,d,e,h,g}$ be, up to $\mathcal{L}$-isotopy of $\mathbb{R}\Sigma_6$, the trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_6$ depicted in b) of Fig. 3.8, where letters i, d, e, h, g_j, for $j = 1, 2, 3, 4$, denote numbers of ovals. Let g be $\sum_{j=1}^{4} g_j$ and let s, k be non-negative integer numbers. Then, there exist real algebraic trigonal curves in Σ_6 realizing the real schemes $\eta_{i,d,e,h,g}$ for all i, d, e, h, g such that $0 \leq g \leq s$, $0 \leq i+d+e+h \leq k$, where s, k are the following:*

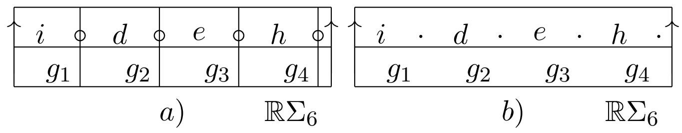


Figure 3.8: $a)$ The union of a trigonal $\mathcal{L}$-scheme and four fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_6$: $\tilde{\eta}_{i,d,e,h,g}$. $b)$ A nodal trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_6$: $\eta_{i,d,e,h,g}$.

(1) $s + k = 12$ *with* $s \in \{6, 10\}$,

(2) $s + k = 10$ *with* $s \in \{5, 9\}$,

(3) $s + k = 8$ *with* $s \in \{0, 4, 6, 8\}$,

In particular, such real algebraic trigonal curves are of type I for $g = s$ *and* $i + d + e + h = k$. *Moreover, there exist real algebraic trigonal curves of type I in* Σ_6 *realizing* $\eta_{i,d,e,h,g}$ *for*

(4) $i + d + e + h + g = 8$ *with* $g = 0$;

(5) $i + d + e + h + g = 6$ *with* $g \in \{1, 3, 5\}$,

(6) $i + d + e + h + g = 4$ *with* $g \in \{2, 4\}$.

Proof. Thanks to Theorems 2.4.3, 2.4.5, if the real graphs associated to $\eta_{i,d,e,h,g}$ are completable in degree 6 to a real trigonal graph of type I (resp. II), then there exist real algebraic trigonal curves of type I (resp. type II) realizing $\eta_{i,d,e,h,g}$.
Let $\tilde{\eta}_{i,d,e,h,g}$ be, up to $\mathcal{L}$-isotopy of $\mathbb{R}\Sigma_6$, the union of a trigonal $\mathcal{L}$-scheme with four fibers of $\mathcal{L}$ on $\mathbb{R}\Sigma_6$ as depicted in $a)$ of Fig. 3.8. Remark that, for i, d, e, h, g as listed in $(1) - (6)$ above, the existence of real trigonal graphs of degree 6 and type I (resp. type II) completing the real graph associated to $\eta_{i,d,e,h,g}$ is equivalent to the existence of those of type I (resp. type II) associated to $\tilde{\eta}_{i,d,e,h,g}$ (see [Ore03]).
We can glue 6 cubic trigonal graphs in such a way that we obtain real trigonal graphs of degree 6 which complete the real graph associated to $\tilde{\eta}_{i,d,e,h,g}$ where i, d, e, h, g are such that $0 \leq g \leq s$ and $0 \leq i + d + e + h \leq k$, for s, k as listed in $(1) - (3)$ above. Type I completions of the real graph associated to $\tilde{\eta}_{i,d,e,h,g}$, for values listed in $(4) - (6)$ above, are pictured in Fig. 3.9. □

Proposition 3.4.4. *There exist real algebraic trigonal curves of type I in* Σ_6 *realizing the trigonal* $\mathcal{L}$*-schemes respectively depicted in* $a)$ *and* $b)$ *of Fig. 3.10. Moreover, there exists a real algebraic trigonal curve in* Σ_6 *realizing the hyperbolic* $\mathcal{L}$*-scheme depicted in Fig. 3.10* $c)$.

Proof. Thanks to Theorems 2.4.3, 2.4.5, if the real graphs associated to the real schemes in the statement are completable in degree 6 to a real trigonal graph of type I (resp. II), then there exist real algebraic trigonal curves of type I (resp. type II) realizing them.
Respective completions in degree 6 of the real graphs associated to the $\mathcal{L}$-schemes in Fig. 3.10 $a), b)$ and $c)$ are pictured in Fig. 3.10 $d)$, $e)$ and $f)$. Furthermore, the trigonal graphs depicted in Fig. 3.10 $d)$, $e)$ and $f)$ are of type I respectively because they have a unique type I labeling and because of Remark 2.4.8. □

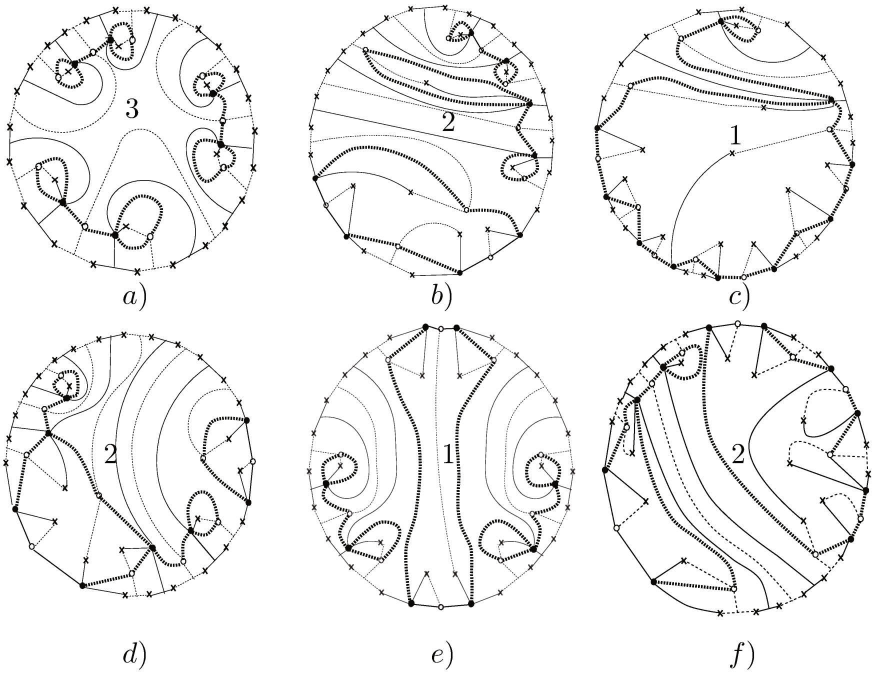


Figure 3.9: Real trigonal graphs of degree 6 and type I.

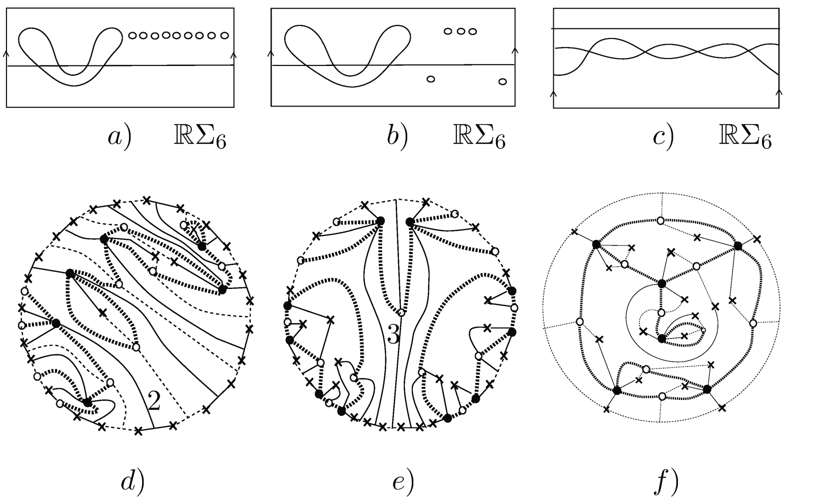


Figure 3.10: Trigonal $\mathcal{L}$-schemes on $\mathbb{R}\Sigma_6$ and the completion of their real graphs in degree 6.

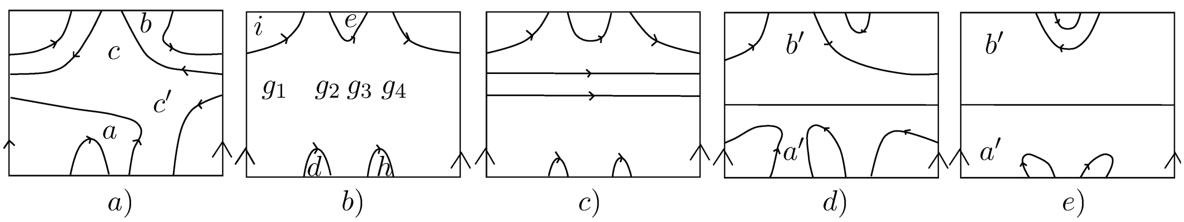


Figure 3.11: $\mathcal{L}$-schemes $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ on $\mathbb{R}\Sigma_2$.

Proposition 3.4.5. *Let η_1, resp. η_2, η_3 and η_4, be a trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_2$, up to $\mathcal{L}$-isotopy of $\mathbb{R}\Sigma_2$, as depicted in Fig. 3.11 a), resp. b), c) and d), where a, b, c, c', resp. i, d, e, h, g, resp. a', b' denote numbers of ovals. Then, such $\mathcal{L}$-scheme is realizable by a non-singular real algebraic curve C_1 (resp. C_2, C_3 and C_4) of bidegree $(3, 4)$ on Σ_2 for a, b, c, c' as listed in Proposition 3.4.1 (resp. i, d, e, h, g as listed in Proposition 3.4.3, resp. $(a', b') = (3, 2)$ and*

$(9,0)$).

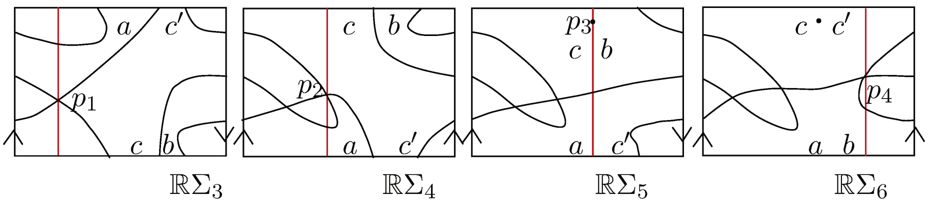


Figure 3.12: Birational transformation of the pair $(\mathbb{R}\Sigma_6, \mathbb{R}\tilde{C}_1)$, from right to left.

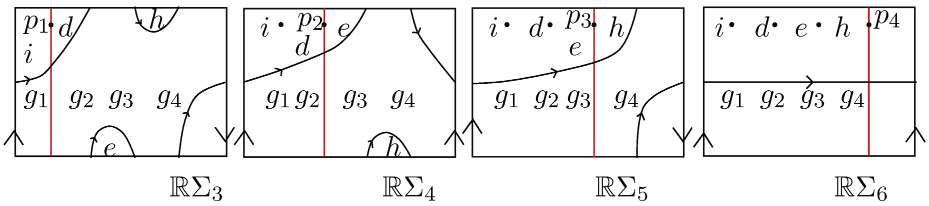


Figure 3.13: Birational transformation of the pair $(\mathbb{R}\Sigma_6, \mathbb{R}\tilde{C}_2)$, from right to left.

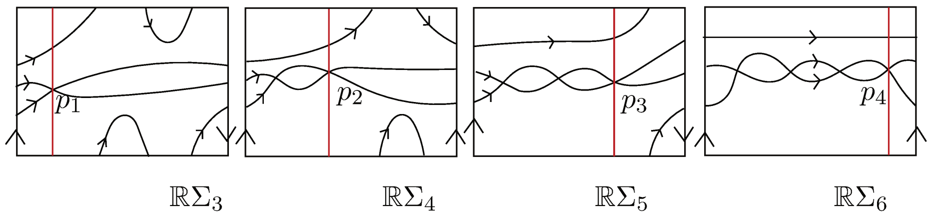


Figure 3.14: Birational transformation of the pair $(\mathbb{R}\Sigma_6, \mathbb{R}\tilde{C}_3)$, from right to left.

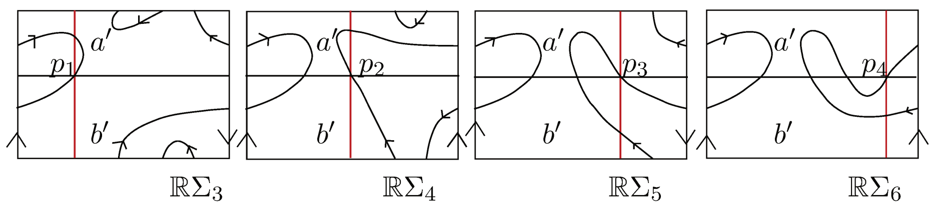


Figure 3.15: Birational transformation of the pair $(\mathbb{R}\Sigma_6, \mathbb{R}\tilde{C}_4)$, from right to left.

Proof. Let us denote by $\tilde{C}_1$ (resp. $\tilde{C}_2$, $\tilde{C}_3$ and $\tilde{C}_4$) any real algebraic trigonal curves in Σ_6 constructed in Propositions 3.4.2 (resp. Proposition 3.4.3, resp. Proposition 3.4.4). Let us consider, as defined in Section 2.3, for each curve $\tilde{C}_j$ the birational transformation $\Xi_j =: \beta_{p_1}^{-1}\beta_{p_2}^{-1}\beta_{p_3}^{-1}\beta_{p_4}^{-1} : (\Sigma_6, \tilde{C}_j) \to (\Sigma_2, C_j)$, where the points p_k, $k = 1,2,3,4$, are real double points such that p_4 belong to $\mathbb{R}\tilde{C}_j$ and p_k, $k = 3,2,1$, to the image of $\mathbb{R}\tilde{C}_j$ via $\beta_{p_{k+1}}^{-1}$. In Fig. 3.12, 3.13, 3.14 and 3.15 we depict in red the fiber of $\mathbb{R}\Sigma_{k+2}$ intersecting the point p_k.

The birational transformation $\beta_{p_2}^{-1}\beta_{p_3}^{-1}\beta_{p_4}^{-1}((\mathbb{R}\Sigma_6, \mathbb{R}\tilde{C}_j))$ is depicted in Fig. 3.12, resp. in Fig. 3.13, Fig. 3.14, Fig. 3.15 from right to left and $\Xi_j((\mathbb{R}\Sigma_6, \mathbb{R}\tilde{C}_j))$ in Fig. 3.11 $a)$ for $j = 1$, resp. in Fig. 3.11 $b), c)$ and $d)$ for $j = 2, 3, 4$.

□

Since we will apply Viro's patchworking method in Section 3.4.2 to construct real algebraic curves of bidegree $(5, 0)$ on Σ_2, in Fig. 3.16 $a), b), c)$ and $d)$ we depict the charts of non-singular real algebraic curves of bidigree $(3, 4)$ in Σ_2 constructed in Proposition 3.4.5. Moreover, performing a coordinates transformation to a curve C_4 with chart as depicted in $d)$ of Fig. 3.16, we obtain a type I real algebraic curve C_5 of bidigree $(3, 4)$ in Σ_2 with chart, resp. real $\mathcal{L}$-scheme, as depicted in $e)$ of Fig. 3.16, resp. as depicted in $e)$ of Fig. 3.11, where a', b' still denote numbers of ovals and $(a', b') = (3, 2)$ or $(9, 0)$.

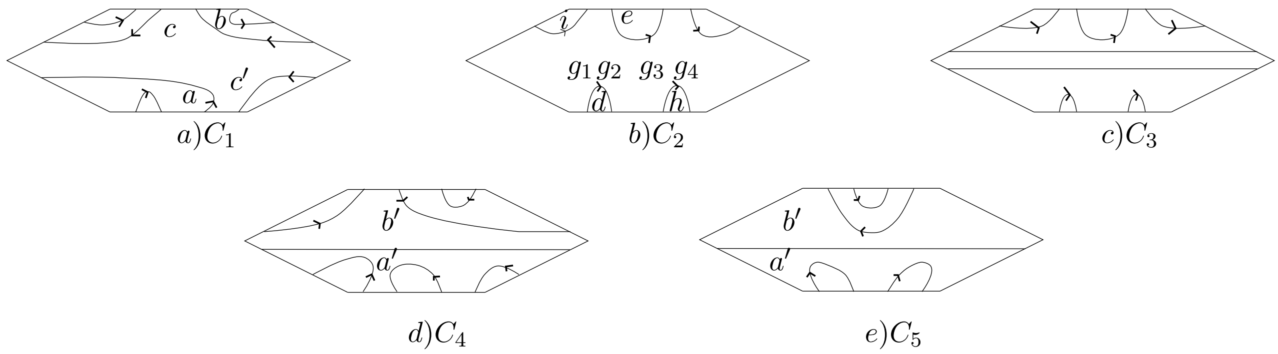


Figure 3.16: Charts of type I curves C_1, C_2, C_3, C_4, C_5 of bidegree $(3, 4)$ in Σ_2.

3.4.2 Final constructions and patchworking

In this Section we end the proof of Theorems 3.1.2, 3.1.3, 3.1.4, 3.1.5. We need Viro's patchworking method. Most of all, we use original Viro's patchworking method which is a tool for constructing real algebraic hypersurfaces with prescribed topology in real toric varieties (Section 2.2.2). Finally, for a particular construction, we use a variant of the patchworking developed by Shustin (Section 2.2.3.1), which exploits the deformation pattern technique and allows to glue charts of polynomials presenting a tangency point with the boundary of the chart.

Remark 3.4.6. *In the proofs of Propositions 3.4.8, 3.4.9, 3.4.10 we construct by patchworking some non-singular real algebraic curves of bidegree* $(5, 0)$ *on* Σ_2. *This, because of the perturbation explained in Proposition 3.3.1, immediately implies the existence of non-singular real algebraic curve of bidegree* $(5, 5)$ *on the quadric ellipsoid.*

Notation 3.4.7. *In Propositions 3.4.8, 3.4.9, 3.4.10 the real schemes marked with the symbol* $^\circ$ *(resp.* **) are realized by a real algebraic curve of type I (resp. type II).*

In the following proposition, we give a construction of real algebraic curves of bidegree $(5, 5)$ on the quadric ellipsoid realizing almost all real schemes listed in Theorems 3.1.2, 3.1.3, 3.1.4 and 3.1.5. The existence of real algebraic curves of bidegree $(5, 5)$ on the quadric ellipsoid realizing the rest of the real schemes listed in the theorems above is proved in Propositions 3.4.9 and 3.4.10.

Proposition 3.4.8. *All the real schemes in the following list are realizable by non-singular real algebraic curves of bidegree* $(5, 5)$ *on the quadric ellipsoid:*

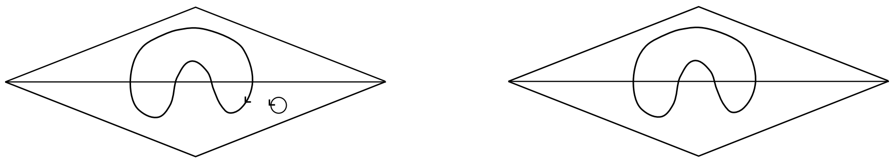


Figure 3.17: Charts and arrangements with respect of the coordinate axis $\{y = 0\}$ of real algebraic curves of bidegree $(2, 0)$ on Σ_2.

(1) all real schemes listed in Theorem 3.1.2 but the real schemes $1 \sqcup \langle 4\rangle \sqcup \langle 10\rangle$ *and* $1 \sqcup \langle 7\rangle \sqcup \langle 7\rangle$*;*

(2) all real schemes listed in Theorem 3.1.3 but the real schemes $\langle 4\rangle \sqcup \langle 10\rangle$ *and* $\langle 7\rangle \sqcup \langle 7\rangle$*;*

(3) all real schemes listed in Theorem 3.1.4 but the real scheme $\langle 4\rangle \sqcup \langle 9\rangle^{\circ}$*;*

(4) all real schemes listed in Theorem 3.1.5 but the real schemes 13°, $1 \sqcup \langle 1\rangle \sqcup \langle 9\rangle^{\circ}$, $1 \sqcup \langle 3\rangle \sqcup \langle 7\rangle^{\circ}$, $\langle 1\rangle \sqcup \langle 4\rangle^{\circ}$, 1 *and* 0.

Proof. For any fixed real four points on the coordinates axis $\{y = 0\}$ there exist real algebraic curves D_1 and D_2 of bidegree $(2, 0)$ on Σ_2 whose charts are respectively as depicted in Fig. 3.17 $a)$ and $b)$ and which intersect $\{y = 0\}$ in the fixed four points. Thanks to Viro's patchworking method and Remark 3.4.6, we realize the real schemes listed in $(1) - (4)$ gluing the polynomials and the charts of a real algebraic curve C_i, with $i = 1, 2, 3, 4, 5$, constructed in Proposition 3.4.5 and at the end of Section 3.4.1, and of a real algebraic curve D_j, with $j = 1, 2$. In Fig. 3.18 it is depicted the patchworking of charts of the C_i of type I with D_1. □

Proposition 3.4.9. *The real schemes* $1 \sqcup \langle 4\rangle \sqcup \langle 10\rangle$, $\langle 4\rangle \sqcup \langle 10\rangle$, $\langle 4\rangle \sqcup \langle 9\rangle^{\circ}$, 13°, $1 \sqcup \langle 1\rangle \sqcup \langle 9\rangle^{\circ}$ *and* $1 \sqcup \langle 3\rangle \sqcup \langle 7\rangle^{\circ}$ *are realizable by non-singular real algebraic curves of bidegree* $(5, 5)$ *on the quadric ellipsoid.*

Proof. For any fixed real ten points on the coordinates axis $\{y = 0\}$ there exist a real algebraic curves $\tilde{C}$ of bidegree $(2, 0)$ on Σ_5 whose charts are as depicted in Fig. 3.19 and which intersects $\{y = 0\}$ in the fixed ten points. For any fixed connected component $\mathcal{O}$ of $\mathbb{R}\tilde{C}$ we can pick four real points, p_1, p_2, p_3, p_4 on it as depicted in Fig. 3.19 $c)$, resp. $d)$. Then, consider the birational transformation $\beta_{p_1}^{-1}\beta_{p_2}^{-1}\beta_{p_3}^{-1}\beta_{p_4}^{-1} : (\Sigma_5, \tilde{C}) \to (\Sigma_1, C')$, as defined in Section 2.3, where we call p_i also the image of p_i via $\beta_{p_j}^{-1}$, $j > i$, $i = 1, 2, 3$. Choose the coordinates axes in $\mathbb{R}\Sigma_1$ such that $\mathbb{R}C'$ has an arrangement as depicted in Fig. 3.20 $b)$, resp. $a)$ where t, s denote numbers of ovals and $t + s = 3$, (the coordinates axes are pictured in red). The charts of C' are as depicted in Fig. 3.21 $c)$, resp. $b)$.

In [OS16], Orevkov and Shustin have constructed some real algebraic curves of bidegree $(4, 0)$ and type I, resp. type II, in Σ_2 whose charts, up to a coordinates change, and arrangements with respect to the coordinates axis $\{x = 0\}$ are as depicted in Fig. 3.21 $a)$ for $(\alpha, \beta, \gamma) = (5, 0, 2), (8, 1, 0)$ and $(4, 1, 0)$, resp. $(7, 0, 1)$, where α, β, γ denote numbers of ovals.

Thanks to Viro's patchworking method and Remark 3.4.6, we realize the real schemes $\langle 4\rangle \sqcup \langle 9\rangle^{\circ}$, 13°, $1 \sqcup \langle 4\rangle \sqcup \langle 10\rangle$, $1 \sqcup \langle 1\rangle \sqcup \langle 9\rangle^{\circ}$, $1 \sqcup \langle 3\rangle \sqcup \langle 7\rangle^{\circ}$ and $\langle 4\rangle \sqcup \langle 10\rangle$ gluing the polynomials and the charts of these latter algebraic curves and of the curves C'; the patchworking of their charts is depicted in Fig. 3.22. □

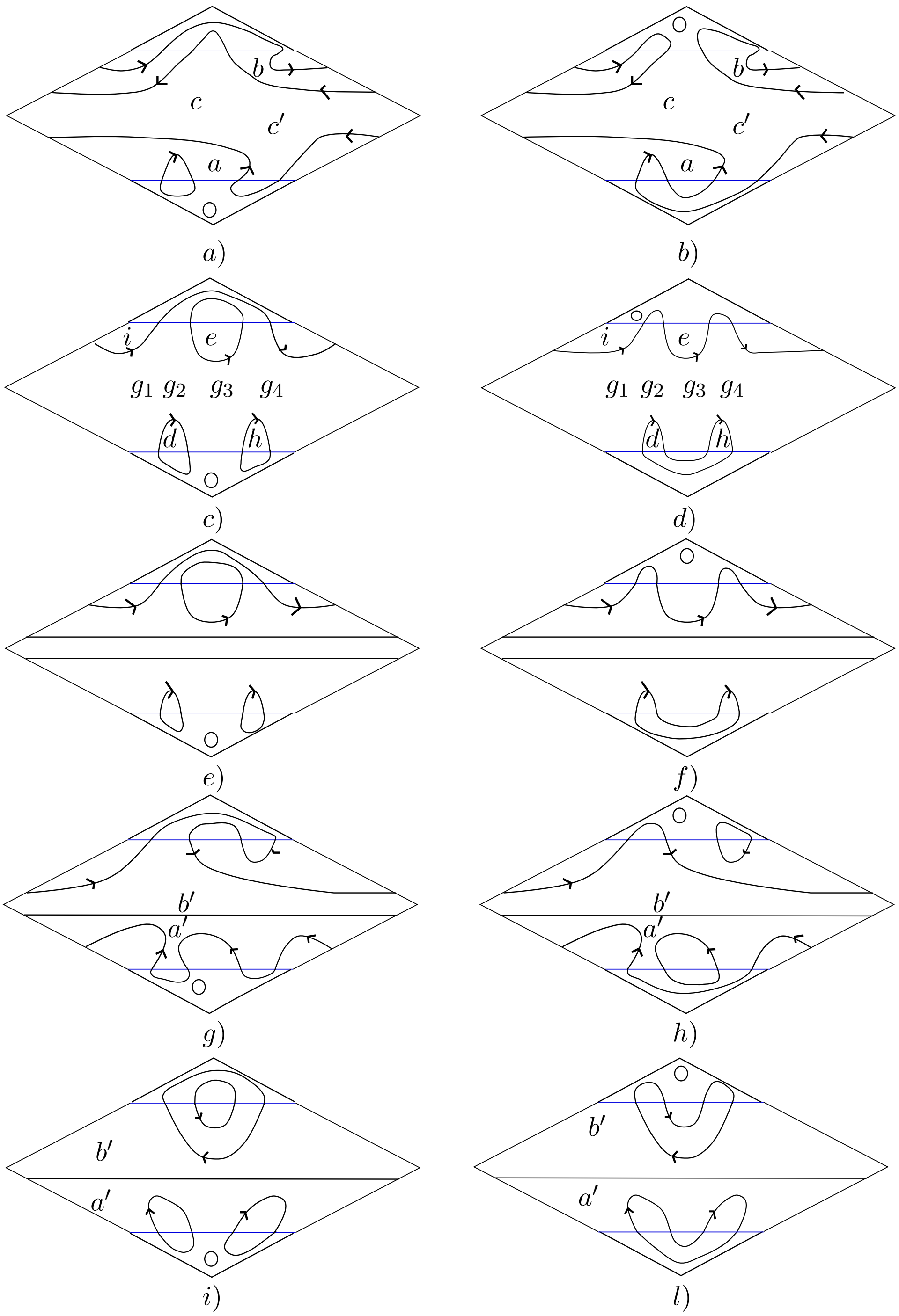


Figure 3.18: Charts of curves of bidegree $(5,0)$ in Σ_2 obtained patchworking the charts of the real algebraic curves C_i from Proposition 3.4.5 and D_1.

Proposition 3.4.10. *The real schemes* $1 \sqcup \langle 7\rangle \sqcup \langle 7\rangle$, $\langle 7\rangle \sqcup \langle 7\rangle$, $\langle 1\rangle \sqcup \langle 4\rangle^{\circ}$, 1 *and* 0 *are realizable by non-singular real algebraic curves of bidegree* $(5,5)$ *on the quadric ellipsoid* X*.*

Proof. First of all, in order to realize the real scheme 0 (resp. 1), we just perturb the union of five real hyperplane sections in $\mathbb{C}P^3$ not intersecting $\mathbb{R}X$ (resp. whose just one intersects $\mathbb{R}X$) to a smooth surface of degree 5. For the realization of the real schemes $\langle 1\rangle \sqcup \langle 4\rangle^{\circ}$ and $1 \sqcup \langle 7\rangle \sqcup \langle 7\rangle$ (resp. $\langle 7\rangle \sqcup \langle 7\rangle$) we give some intermediate constructions, then we apply Viro's (resp. Shustin's) patchworking method.

Let L be any real algebraic line on the complex projective plane. For any fixed five distinct real points on L, we can construct real algebraic plane quintics

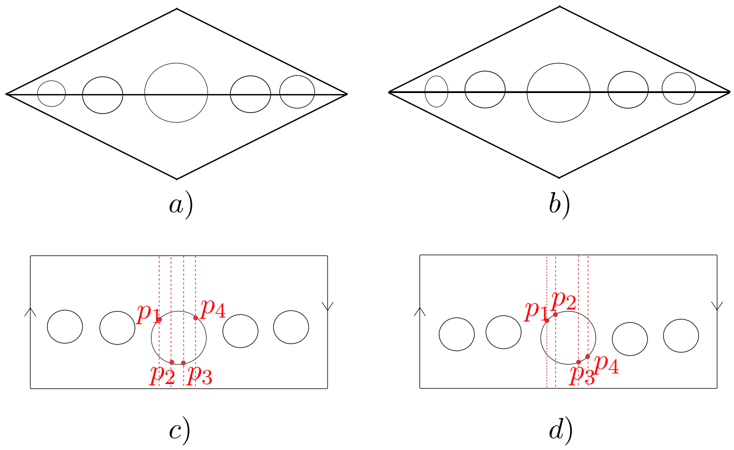


Figure 3.19: Charts and $\mathcal{L}$-schemes of real algebraic maximal curves of bidegree $(2, 0)$ on $\mathbb{R}\Sigma_5$.

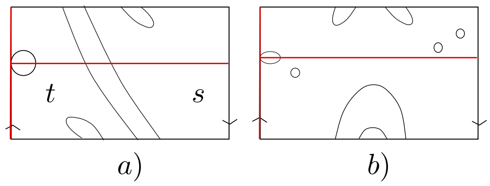


Figure 3.20: $\mathcal{L}$-schemes of real algebraic curves of bidegree $(2, 4)$ on Σ_1.

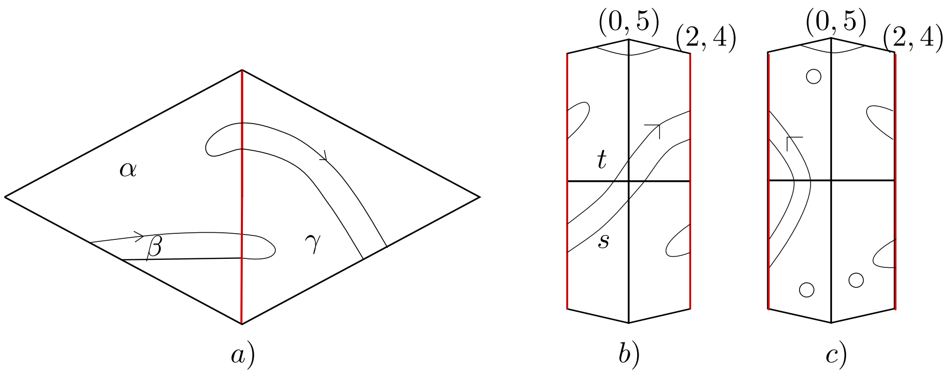


Figure 3.21: $a)$ Charts of real algebraic curves of bidegree $(4, 0)$ on Σ_2. $b), c)$ Charts of real algebraic curves of bidegree $(2, 4)$ on Σ_1.

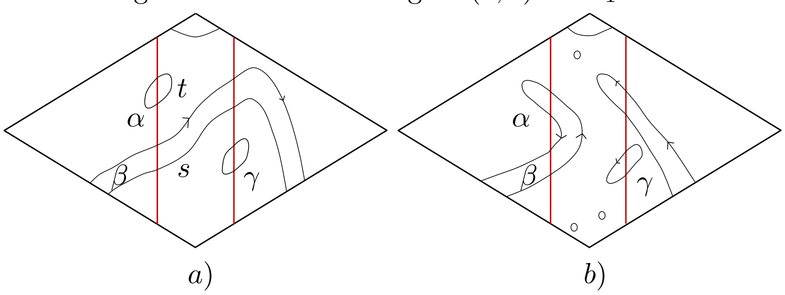


Figure 3.22: Charts of real algebraic curves of bidegree $(5, 0)$ on Σ_2.

passing through such fixed points, obtained as evolving of the singularity N_{16}^2 (see [Vir86]). In particular, there exist non-singular real algebraic quintics R_1, R_2 and R_3 whose charts are respectively as depicted in Fig. 3.23 $a), b)$ and $c)$ with $(p, q) = (6, 0), (0, 6)$, $n = 4$, where p, q and n denote numbers of ovals. Moreover, let

$$P_h(x, y, z) = \sum_{i+j \leq 5} a_{i,j} x^i y^j z^{5-i-j}$$

be a polynomial of a quintic R_h, with $h = 2$, 3, passing through five fixed

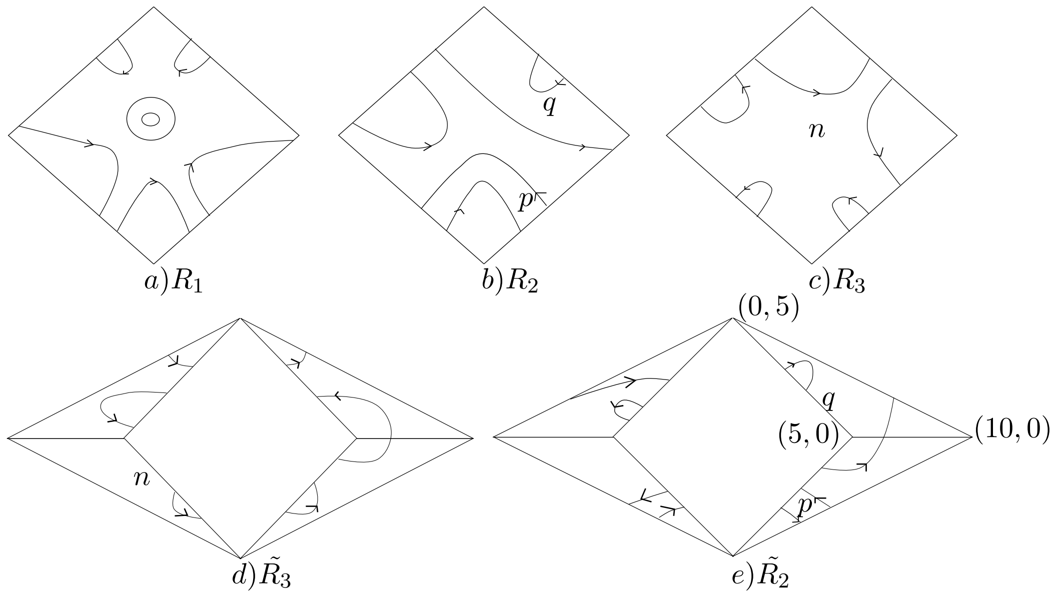
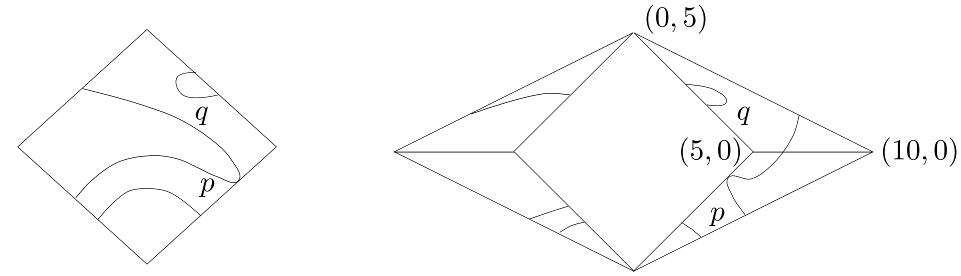
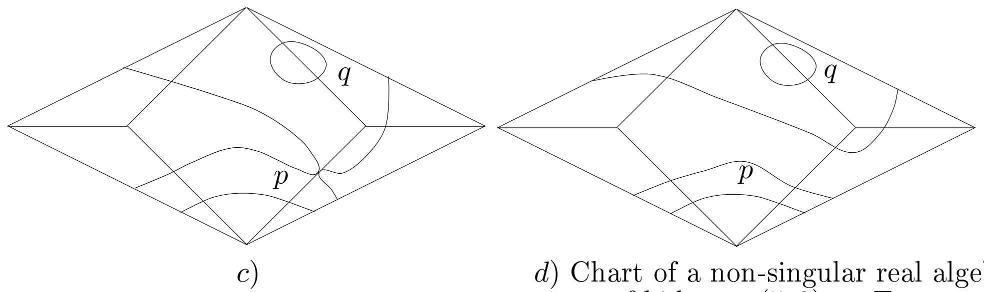


Figure 3.23: Charts of real algebraic curves.



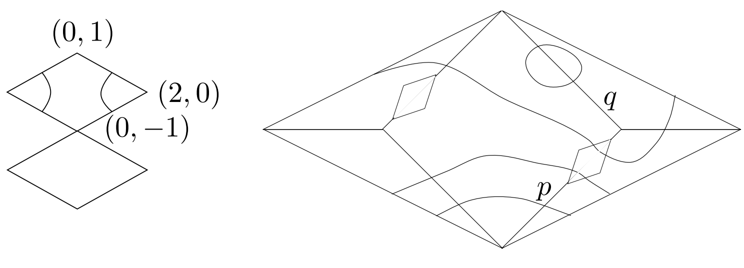
$a)$ Chart of a real algebraic plane quintic. $b)$ Chart of a real algebraic plane curve.



$c)$

$d)$ Chart of a non-singular real algebraic curve of bidegree $(5,0)$ on Σ_2.

Figure 3.24:



$a)$ Deformation pattern.

$b)$ Chart of a non-singular real algebraic curve of bidegree $(5,0)$ on Σ_2.

Figure 3.25:

points on L. Then, applying to the polynomials P_h the transformation $T : P_h(x,y,z) \mapsto P_h(xz, yz, x^2)$ we construct real algebraic plane curves $\tilde{R}_h$ whose polynomials are

$$\tilde{P}_h(x,y,z) = \sum_{i+j\leq 5} a_{i,j} x^{10-i-2j} y^j z^{i+j}$$

and whose charts are as depicted in Fig. 3.23 d) and e) respectively for $n = 4$ and $(p,q) = (0,6), (6,0)$, where n, p and q still denote numbers of ovals. Later on, we use Viro's patchworking theorem, so remark that P_h and $\tilde{P}_h$ have monomials with same coefficient $a_{i,j}$ for $i + j = 5$.
In order to realize the real scheme $\langle 1 \rangle \sqcup \langle 4 \rangle^{\circ}$, resp. $1 \sqcup \langle 7 \rangle \sqcup \langle 7 \rangle$, we apply Viro's patchworking method gluing the polynomials and the charts of the real plane quintic R_1, resp. R_2 with $(p,q) = (0,6)$, with the real algebraic curve $\tilde{R}_3$, resp. $\tilde{R}_2$ with $(p,q) = (6,0)$.
Finally, the construction of a real algebraic curve of bidegree $(5,5)$ on the quadric ellipsoid realizing the real scheme $\langle 7 \rangle \sqcup \langle 7 \rangle$ requires a variant of the patchworking theorem due to Shustin (Section 2.2.3.1). For any four fixed real points on L, there exists a real algebraic plane quintic passing through three such points and tangent to L at the remaining fixed point and whose chart is as depicted in Fig. 3.24 a), where $(p,q) = (6,0)$ and $(0,6)$ and p, q denote numbers of ovals. Moreover, there exists a real algebraic plane curve with chart as depicted in b) of Fig. 3.24, constructed from the quintic via the transformation T above. Gluing the charts of those two real algebraic plane curves respectively with $(p,q) = (6,0)$ and $(p,q) = (0,6)$, as depicted in c) of Fig. 3.24 (where p and q are both equal to 6), we do not obtain a chart of a polynomial. But, the variant of the patchworking method developed by Shustin allows us to replace a neighborhood of the two tangency points with a *deformation pattern* (as depicted in a) of Fig. 3.25); see b) of Fig. 3.25. At the end, we obtain a chart of a real algebraic curve of bidegree $(5,0)$ in Σ_2 as depicted in d) of Fig. 3.24, with $p = 6$ and $q = 6$. $\square$

Chapter 4

Real algebraic curves on real minimal del Pezzo surfaces of degree 1

4.1 Introduction

Let Y be $\mathbb{C}P^2$ blown up at eight points in generic position; then, the surface Y is a del Pezzo surface of degree 1 (see [BCC+08, pag. 289-312], [Dol12, Chapter 8]). The anti-bicanonical map $\psi : Y \to \mathbb{C}P^3$ exhibits Y as a double ramified cover of an irreducible singular quadric Q in $\mathbb{C}P^3$; the branch locus of ψ consists of the node V of Q and a non-singular cubic section $\tilde{S}$ on Q disjoint from V. Conversely, any such double covering is a del Pezzo surface of degree 1.
By construction, the first Chern class $c_1(Y)$ is the pull back via ψ of the class of a line on Q ([DIK00]). Let us consider the standard real structure of $\mathbb{C}P^3$. Suppose that the quadric Q and $\tilde{S}$ are real. Let Q_1 and Q_2 be two distinct disjoint unions of connected components of $\mathbb{R}Q \setminus (\mathbb{R}\tilde{S} \cup \{V\})$ such that each Q_i is bounded by $\mathbb{R}\tilde{S} \cup \{V\}$. There exist two lifts to Y of the real structure of Q via the double cover ψ and the real part of Y is the double of one of the Q_i's. Let c be a lifting to Y of the standard real structure on Q, then (Y, c) is $\mathbb{R}$-minimal if and only if Q is a quadratic cone, the cubic section $\tilde{S}$ is real and maximal, and $\mathbb{R}Y := fix(c)$ is homeomorphic to $\mathbb{R}P^2 \bigsqcup_{j=1}^{4} S^2$; see [DK02], [DIK00] and Fig. 4.1.

Notation 4.1.1. *From now, up to the end of this chapter the symbol Q denotes the quadratic cone with equation $X^2 + Y^2 - Z^2 = 0$ in $\mathbb{C}P^3$. Moreover, the real part of Q will be depicted as a quadrangle whose opposite sides are identified in a suitable way and the horizontal sides represent the node $V := [0 : 0 : 0 : 1]$ of Q.*

For any two distinct real maximal cubic sections on Q, one may obtain non-isomorphic real minimal del Pezzo surfaces of degree 1 via double ramified covers; nevertheless, all real minimal del Pezzo surface of degree 1 are equivalent up to equivariant deformation; see [DK02].
Let $c_* : H_2(Y; \mathbb{Z}) \to H_2(Y; \mathbb{Z})$ be the group homomorphism induced by the real structure c on Y, and let $H_2^-(Y; \mathbb{Z})$ be the (-1)-eigenspace of c_*. If (Y, c) is minimal, then $H_2^-(Y; \mathbb{Z})$ is generated by $c_1(Y)$ ([BCC+08, pag. 289-312]), and any real algebraic curve in Y realizes $kc_1(Y)$, where k is some non-negative integer.

Definition 4.1.2. *Let Y be a real minimal del Pezzo surface of degree 1 and*

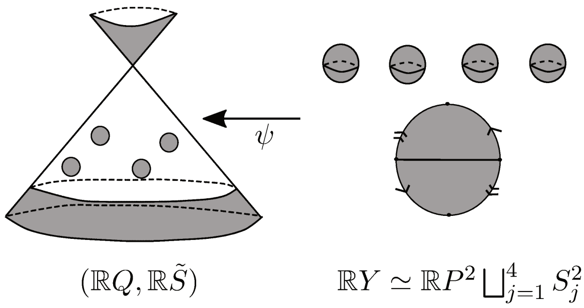


Figure 4.1: Real part of the ramified double cover map.

let $B \subset Y$ be a non-singular real algebraic curve. Then, we say that B has class k on Y if B realizes $kc_1(Y)$ in $H_2(Y;\mathbb{Z})$.

In this chapter, we are interested in the classification up to homeomorphism of the pairs $(\mathbb{R}Y, \mathbb{R}B)$, where B is a non-singular real algebraic curve of class k in a real minimal del Pezzo surface Y of degree 1. The first homology group $H_1(\mathbb{R}Y;\mathbb{Z}/2\mathbb{Z})$ has two homology classes, and we call *ovals* (resp. *pseudo-lines*) the connected components of $\mathbb{R}B$ realizing the trivial class (resp. the non-trivial class) in $H_1(\mathbb{R}Y;\mathbb{Z}/2\mathbb{Z})$.

4.1.1 Positive and negative connected components of $\mathbb{R}Y$

Let $\tilde{S}$ be a real maximal cubic section on Q. The two halves of $\tilde{S}\backslash\mathbb{R}\tilde{S}$ induce two opposite orientations on $\mathbb{R}\tilde{S}$, called *complex orientations*. Independently from the choice of a complex orientation on $\mathbb{R}\tilde{S}$, we can distinguish the connected components of $\mathbb{R}\tilde{S}$ on $\mathbb{R}Q$ in the following way. Let us consider $Q \setminus \{V\}$. There are four connected components, called *ovals*, of $\mathbb{R}\tilde{S}$ realizing the trivial class in $H_1(\mathbb{R}Q \setminus \{V\};\mathbb{Z}/2\mathbb{Z})$; while the connected component of $\mathbb{R}\tilde{S}$ realizing the non-trivial class in $H_1(\mathbb{R}Q \setminus \{V\};\mathbb{Z}/2\mathbb{Z})$, is called *long-component*. If the union of an oval and the long-component of $\mathbb{R}\tilde{S}$ in $\mathbb{R}Q\setminus\{V\}$ bounds an oriented surface, the oval is called *positive*; otherwise *negative*.
Each oval of $\mathbb{R}\tilde{S}$ is either positive or negative, independently from the choice

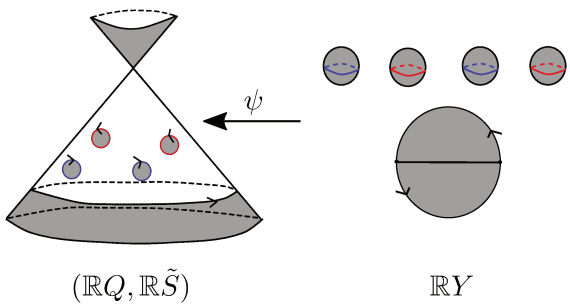


Figure 4.2: Positive and negative connected components of $\mathbb{R}Y$.

of a complex orientation on $\mathbb{R}\tilde{S}$; moreover, two ovals are positive and two are negative, and they alternate because of Fiedler's Theorem ([Fie83]). Now, we can differentiate the connected components of $\mathbb{R}Y$ homeomorphic to S^2 as positive and negative via the double ramified cover map ψ; see Fig. 4.2, where the preimage of negative and positive ovals, is depicted respectively in red and blue.

Notation 4.1.3. *Let Y be a real minimal del Pezzo surface of degree 1. We denote the connected components of $\mathbb{R}Y$ with Y_0, Y_1, Y_2, Y_3, Y_4, where Y_0 is homeomorphic to $\mathbb{R}P^2$ and Y_j, with $j \geq 1$, is homeomorphic to S^2. Moreover,* *we use the convention that the connected components Y_1 and Y_2 are positive and Y_3 and Y_4 negative.*

4.1.2 Real schemes

Given a collection $\bigsqcup_{h=1}^{l} O_h$ of l disjoint circles embedded in $\mathbb{R}P^2$, resp. S^2, the arrangement of the pair $(\mathbb{R}P^2, \bigsqcup_{h=1}^{l} O_h)$, resp. the arrangement of the pair $(S^2, \bigsqcup_{h=1}^{l} O_h)$, is encoded as in Section 2.1. Let Y be a real minimal del Pezzo surface of degree 1. We encode the arrangement of a given collection $\bigsqcup_{h=1}^{l} O_h$ of l disjoint circles embedded in $\mathbb{R}Y$ as follows. Let $\mathcal{S}_j$ be the codification of the arrangement of the embedded circles in Y_j with $0 \leq j \leq 4$. We say that the pair $(\mathbb{R}Y, \bigsqcup_{h=1}^{l} O_h)$ realizes $\mathcal{S}_0 \mid \mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$.

Definition 4.1.4. *We say that $\mathcal{S}_0 \mid \mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$ is a **realizable real scheme in class** k (resp. **realizable coarse real scheme in class** k), if there exists a real minimal del Pezzo surface Y of degree 1, and a real algebraic curve $B \subset Y$ of class k, such that the pair $(\mathbb{R}Y, \mathbb{R}B)$ realizes $\mathcal{S}_0 \mid \mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$ (resp. if there exists a permutation $\sigma \in S_4$ such that the pair $(\mathbb{R}Y, \mathbb{R}B)$ realizes $\mathcal{S}_0 \mid \mathcal{S}_{\sigma(1)} : \mathcal{S}_{\sigma(2)} : \mathcal{S}_{\sigma(3)} : \mathcal{S}_{\sigma(4)}$).*

Example 4.1.5. *Suppose that the (coarse) real scheme depicted in Fig. 4.3 is realizable in class 3 in a real minimal del Pezzo surface Y of degree 1. One can say that the coarse real scheme $\mathcal{J} \sqcup \langle 1 \rangle \mid 1 : 1 : 0 : 0$ is realizable in class 3 in Y; and the real scheme $\mathcal{J} \sqcup \langle 1 \rangle \mid 0 : 1 : 1 : 0$ is realizable in class 3 in Y; .*

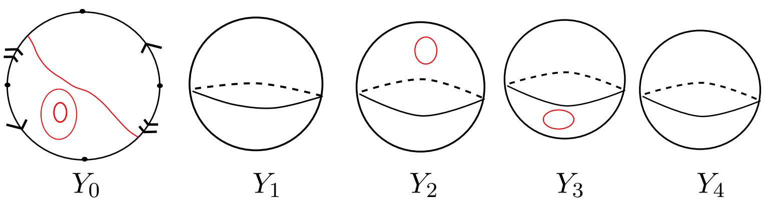


Figure 4.3: Example of (coarse) real scheme in class 3.

4.1.3 Main results

Theorem 4.1.6. *Let B be a real algebraic curve of class k in a real minimal del Pezzo surface of degree 1, with $k \in \{1, 2, 3\}$. Then B realizes one of the following coarse real schemes:*

(1) If $k = 1$:

- *(1) $\mathcal{J} \sqcup 1 \mid 0 : 0 : 0 : 0$;*
- *(2) $\mathcal{J} \mid 1 : 0 : 0 : 0$;*
- *(3) $\mathcal{J} \mid 0 : 0 : 0 : 0$.*

(2) If $k = 2$:

- *(1) $\alpha \sqcup \langle \beta \rangle \mid \gamma : \delta : 0 : 0$, with $0 \leq \alpha + \beta + \gamma + \delta \leq 2$, and $0 \leq \gamma \leq \delta$;*
- *(2) $0 \mid \alpha \sqcup \langle \beta \rangle : \gamma : \delta : 0$, with $0 \leq \alpha + \beta + \gamma + \delta \leq 2$, $0 \leq \gamma \leq \delta$ and $0 \leq \alpha \leq \beta$;*
- *(3) $\langle\langle 1 \rangle\rangle \mid 0 : 0 : 0 : 0$.*

(3) If $k = 3$:

(1) $\mathcal{J} \sqcup \alpha \sqcup \langle\beta\rangle \mid \gamma : \delta : \varepsilon : 0$, *with* $0 \leq \alpha+\beta+\gamma+\delta+\varepsilon \leq 3$ *and* $0 \leq \gamma \leq \delta \leq \varepsilon$*;*

(2) $\mathcal{J} \sqcup \alpha \sqcup \langle\langle\beta\rangle\rangle \mid \gamma : 0 : 0 : 0$, *with* $0 \leq \alpha+\beta+\gamma \leq 2$ *and* $\beta \neq 0$*;*

(3) $\mathcal{J} \mid \alpha \sqcup \langle\beta\rangle \sqcup \langle\gamma\rangle : \delta : \varepsilon : 0$, *with* $0 \leq \alpha+\beta+\gamma+\delta+\varepsilon+\zeta+\eta \leq 2$ *and* $0 \leq \beta \leq \gamma$, $0 \leq \delta \leq \varepsilon$*;*

(4) $\mathcal{J} \sqcup \langle\langle\langle 1\rangle\rangle\rangle \mid 0 : 0 : 0 : 0$*;*

(5) $\mathcal{J} \sqcup \langle 1\rangle \sqcup \langle 1\rangle \mid 0 : 0 : 0 : 0$*;*

(6) $\mathcal{J} \sqcup \langle 1 \sqcup \langle 1\rangle\rangle \mid 0 : 0 : 0 : 0$*;*

(7) $\mathcal{J} \sqcup 1 \mid 1 \sqcup \langle 1\rangle : 0 : 0 : 0$*;*

(8) $\mathcal{J} \mid 1 : 1 : 1 : 1$*;*

(9) $\mathcal{J} \mid 0 : 0 : 0 : 0$.

Moreover, let $\mathcal{S}_0 \,|\mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$ *be any real scheme in the above list. Then, the real scheme* $\mathcal{S}_0 \mid \mathcal{S}_{\sigma(1)} : \mathcal{S}_{\sigma(2)} : \mathcal{S}_{\sigma(3)} : \mathcal{S}_{\sigma(4)}$*, for any permutation* $\sigma \in S_4$*, is realizable by a real algebraic curve of class* k *in a real minimal del Pezzo surface of degree* 1*.*

4.2 Obstructions

The number of pseudo-lines (see Section 4.1) of a real algebraic curve of class k in a real minimal del Pezzo surface Y of degree 1, is determined by k.

Proposition 4.2.1. *Let* B *be a non-singular real algebraic curve of class* k *in a real minimal del Pezzo surface* Y *of degree* 1*. Then, the real scheme realized by* $\mathbb{R}B$ *has one and only one pseudo-line if* $k \equiv 1 \pmod 2$ *and no pseudo-lines otherwise.*

Proof. Let k be odd (resp. even). Since the value modulo 2 of the intersection form on $H_2^-(Y;\mathbb{Z})$ descends on $H_1(\mathbb{R}Y;\mathbb{Z}/2\mathbb{Z}) \simeq H_1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$, it follows that $\mathbb{R}B$ has an odd (resp. even) number of pseudo-lines. Moreover, the real part of B has at most one pseudo-line since any two pseudo-lines meet in at least one point. □

Directly from Harnack-Klein's inequality and Proposition 4.2.1 we obtain the following statement.

Proposition 4.2.2. *Let* B *be a non-singular real algebraic curve of class* k *in a real minimal del Pezzo surface* Y *of degree* 1*. Then, the number* l *of connected components of* $\mathbb{R}B$ *is bounded as follows:*

$$\varepsilon \leq l \leq \frac{k(k-1)}{2} + 2,$$

where $\varepsilon \in \{0,1\}$ *is such that* $\varepsilon \equiv k \pmod 2$.

Proof. The right inequality follows from Harnack-Klein's inequality and the adjunction formula; while the left one follows from Proposition 4.2.1. □

Theorem 4.1.6 states that the system of obstructions given in Propositions 4.2.2 and 4.2.1 is complete for real algebraic curves of class 1, 2 and 3. In Section 4.3 we realize all the (coarse) real schemes listed in Theorem 4.1.6. The next Proposition provides an example of additional obstructions on real schemes realized by non-singular real algebraic curves of class k in a real minimal del Pezzo surface of degree 1, with $k > 3$.

Proposition 4.2.3. *Let B be a non-singular real algebraic curve of class $k = 2s+\varepsilon$ in a real minimal del Pezzo surface Y of degree 1, where $\varepsilon \in \{0,1\}$ and $s \in \mathbb{Z}_{\geq 1}$. Suppose that all connected components of $\mathbb{R}B$ lie on Y_0 and on t of the Y_j's, for $j = 1,2,3,4$. Suppose that N_h, with $h = 1,2,3$, are three nests of depth i_h of $\mathbb{R}B$. Assume that $i_1 \leq i_2$ and N_1, N_2 form a disjoint pair of nests in Y_0; while N_3 lies on some Y_j, where $j \neq 0$. Then, we have the following restrictions on the depths of the nests:*

$$i_1 + i_2 \leq 3s + \varepsilon - t; \tag{4.1}$$

$$i_2 + i_3 \leq 3s + \varepsilon - (t-1). \tag{4.2}$$

Proof. For any given collection $\mathcal{P}$ of 6 distinct points on Y, there exists an algebraic curve T of class 3 on Y passing through $\mathcal{P}$. If all points of $\mathcal{P}$ are real and such that each connected component of $\mathbb{R}Y$ contains at least one point of $\mathcal{P}$, then T is real and $\mathbb{R}T$ has exactly one connected component on each connected component of $\mathbb{R}Y$.
Let us choose such a collection $\mathcal{P}$ in the following way. On each boundary of the two disks in $Y_0 \setminus \bigsqcup_{h=1}^{2} N_h$, pick a point. Moreover, pick a point on every connected component Y_j, with $j \geq 1$, such that the point belongs to $\mathbb{R}B$ any time the real algebraic curve has at least one oval on Y_j. Then, there exists a non-singular real algebraic curve T of class 3 passing through $\mathcal{P}$. Thus, the number of real intersection points of $\mathbb{R}B$ with $\mathbb{R}T$ is at least $2(i_1+i_2+t)+\varepsilon$. Therefore, inequality (4.1) follows directly from the fact that the intersection number $B \circ T = 3(2s+\varepsilon)$ is greater or equal to the number of real intersection points of B with T.
The proof of (4.2) is similar to the previous one. □

Example 4.2.4 (Application of Proposition 4.2.3)**.** *Let us apply inequality* (1) *of Proposition 4.2.3 to show that the real scheme $\mathcal{S} := \langle 1 \rangle \sqcup \langle 1 \rangle \mid 2:1:1:0$ in class 4 on $\mathbb{R}Y$ is unrealizable; see Fig. 4.4. Let us choose a configuration $\mathcal{P}$ of 6 real points $p_1,..,p_6$ as depicted in Fig. 4.4; then, there exists a non-singular real algebraic curve T passing through $\mathcal{P}$ as depicted in blue in Fig. 4.4. The real scheme $\mathcal{S}$ has two nests of depth 2 on Y_0 and ovals on Y_1, Y_2 and Y_3; applying inequality* (1) *it follows that 4 has to be less or equal to 3; this contradiction implies that the real scheme $\mathcal{S}$ is unrealizable in class 4.*

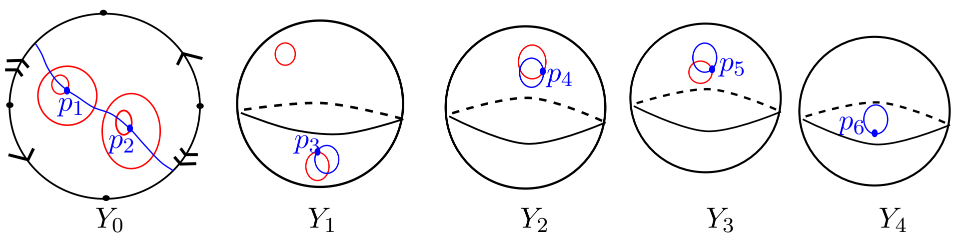


Figure 4.4: Example of unrealizable real scheme in class 4.

Remark 4.2.5. *Exploiting Welschinger-type invariants on Y (see [Bru18]), other topological obstructions may be found for real algebraic curves of class $k \geq 4$ on Y. We give an example of such technique of topological obstruction in Proposition 5.2.5 for real schemes of real algebraic curves of class d on a 4-spheres real del Pezzo surface of degree 2 (see Chapter 5).*

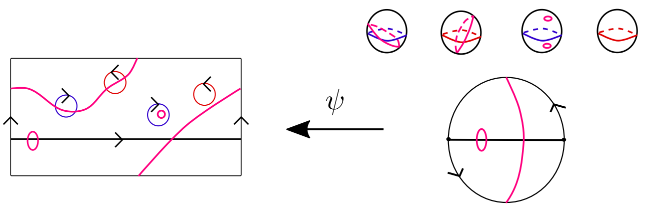
4.3 Constructions

4.3.1 Constructions from the quadratic cone

Let $Bl_V : \Sigma_2 \to Q$ be the blow-up of the quadratic cone Q at the node V; see Section 2.3. The fibration of the second Hirzebruch surface Σ_2 is the extension of the projection from the blown-up point to a hyperplane section which does not pass through the blown-up point. The group $H_2(\Sigma_2;\mathbb{Z})$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ and is generated by the classes $[B_2]$ and $[F_2]$. Let $\tilde{C}$ be an algebraic curve on Q and C the strict transform of $\tilde{C}$ via Bl_V^{-1} in Σ_2. The curve C is said to be of bidegree (k,l) if it realizes the homology class $k[B_2]+l[F_2]$ in $H_2(\Sigma_2;\mathbb{Z})$.

Definition 4.3.1. *Let k and l be two non-negative integers. We say that an algebraic curve $\tilde{C}$ on Q has bidegree (k,l) if $Bl_V^{-1}(\tilde{C})$ has bidegree (k,l) in Σ_2.*

Let $\tilde{S}$ be a real maximal curve of bidegree $(3,0)$ on Q and let Y be a real minimal del Pezzo surface of degree 1 constructed via a double cover $\psi : Y \to Q$ ramified along $\tilde{S}$. Let T be any real algebraic curve of bidegree (k,ε), with $\varepsilon\in\{0,1\}$. From the arrangement of the triplet $(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T)$, we recover the real scheme realized by the real algebraic curve $\psi^{-1}(T)$ of class $2k+\varepsilon$ in Y. See as example Fig. 4.5: on the left, the arrangement of a triplet $(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T)$; on the right, the induced real scheme $\mathcal{J}\ \sqcup\ 1\mid 1:0:1:2$ on $\mathbb{R}Y$.

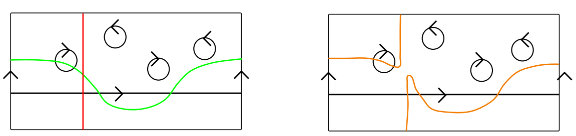


$(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T)$ $\qquad$ $\mathcal{J}\ \sqcup\ 1\mid 1:0:1:2$

Figure 4.5: Preimage via ψ of a particular triplet $(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T)$.

Remark 4.3.2. *Let $\tilde{S}$ and T_1 be non-singular real algebraic curves on Q respectively of bidegree $(3,0)$ and $(1,0)$. Let F be any real line on Q. By a small perturbation one can perturb $T_1\cup F$ in order to construct a bidegree $(1,1)$ non-singular real algebraic curve T_2 on Q. In particular, from the arrangement of the triplet $(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T_1\cup\mathbb{R}F)$ one can recover the arrangement of the triplet $(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T_2)$.*

Example 4.3.3 (Application of Remark 4.3.2)**.** *Let $\tilde{S}$, T_1 and F be non-singular real algebraic curves on Q respectively of bidegree $(3,0)$, $(1,0)$ and $(0,1)$. Let the topological type of $(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T_1\cup\mathbb{R}F)$ be as depicted in Fig. 4.6, on the left. One can perturb $T_1\cup F$ in order to construct a bidegree $(1,1)$ non-singular real algebraic curve T_2 on Q such that the arrangement of $(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T_2)$ is as depicted on the right of Fig. 4.6.*



$(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T_1\cup\mathbb{R}F)$ $\qquad$ $(\mathbb{R}Q,\mathbb{R}\tilde{S},\mathbb{R}T_2)$

Figure 4.6: Application of construction in Remark 4.3.2.

4.3.2 Small perturbations method on Q

In Proposition 4.3.4, we realize some of the real schemes listed in Theorem 4.1.6.

Proposition 4.3.4. *Let $\mathcal{S}_0 \mid \mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$ be any of the coarse real schemes listed below. Then, each real scheme $\mathcal{S}_0 \mid \mathcal{S}_{\sigma(1)} : \mathcal{S}_{\sigma(2)} : \mathcal{S}_{\sigma(3)} : \mathcal{S}_{\sigma(4)}$ is realizable, for any permutation $\sigma \in S_4$, by a real algebraic curve of class k in a real minimal del Pezzo surface of degree 1, with $k \in \{1, 2, 3\}$.*

(1) For $k = 1$:

- *(1)* $\mathcal{J} \sqcup 1 \mid 0 : 0 : 0 : 0$;
- *(2)* $\mathcal{J} \mid 1 : 0 : 0 : 0$;
- *(3)* $\mathcal{J} \mid \ : 0 : 0 : 0 : 0$.

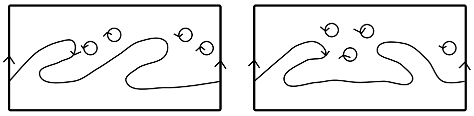
(2) For $k = 2$:

- *(1)* $\alpha \sqcup \langle\beta\rangle \mid \gamma : \delta : 0 : 0$, *with* $0 \leq \alpha + \beta + \gamma + \delta \leq 2$, *and* $0 \leq \gamma \leq \delta$;
- *(2)* $\langle\langle 1 \rangle\rangle \mid 0 : 0 : 0 : 0$.

(3) For $k = 3$:

- *(1)* $\mathcal{J} \sqcup \alpha \sqcup \langle\beta\rangle \mid \gamma : \delta : \varepsilon : 0$, *with* $0 \leq \alpha + \beta + \gamma + \delta + \varepsilon \leq 3$, *and* $0 \leq \gamma \leq \delta \leq \varepsilon < 3$;
- *(2)* $\mathcal{J} \sqcup \alpha \sqcup \langle\langle\beta\rangle\rangle \mid \gamma : 0 : 0 : 0$, *with* $0 \leq \alpha + \beta + \gamma \leq 2$ *and* $\beta \neq 0$;
- *(3)* $\mathcal{J} \sqcup \langle\langle\langle 1 \rangle\rangle\rangle \mid 0 : 0 : 0 : 0$;
- *(4)* $\mathcal{J} \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \mid 0 : 0 : 0 : 0$;
- *(5)* $\mathcal{J} \sqcup \langle 1 \sqcup \langle 1 \rangle\rangle \mid 0 : 0 : 0 : 0$.

Proof. There exists a real algebraic maximal curve $\tilde{S}$ of bidegree $(3, 0)$ on Q with real part as depicted in $a)$ of Fig. 4.7 (resp. $b)$ of Fig. 4.7). Any real line F on Q lifts to a real algebraic curve of class 1 on a real minimal del Pezzo surface of degree 1; in Fig. 4.8 from the depicted real lines on Q, one recovers all real schemes listed in (1) above.
For any two lines F_1, F_2 on Q there exists an hyperplane section $H \subset \mathbb{C}P^3$ passing through V such that $Q \cap H = F_1 \cup F_2$. Moving slightly H, one constructs a real algebraic curve T_1 of bidegree $(1, 0)$ on Q such that from the arrangement of the triplet $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}F_1 \cup \mathbb{R}F_2)$ one can recover the arrangement of the triplet $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}T_1)$. See as example $a)$ of Fig. 4.9, where we have depicted a $(0, 2)$ real algebraic curve as intersection of Q with an hyperplane section H passing through V; resp. in $b)$ of Fig. 4.9, moving slightly H, we have constructed a $(1, 0)$ real algebraic curve on Q. Considering all the possible arrangements of $\tilde{S}$ and two lines, one can construct real algebraic curves T_1 of bidegree $(1, 0)$ from which we deduce that all the real schemes listed in (2) are realizable by real algebraic curves of class 2 in a real minimal del Pezzo surface of degree 1. Moreover, via the construction explained in Remark 4.3.2 starting from real algebraic curves T_1, one can also construct real algebraic curves T_2 of bidegree $(1, 1)$ from which we deduce that all the real schemes listed in (3) are realizable by real algebraic curves of class 3 in a real minimal del Pezzo surface of degree 1. As example, from the arrangement of the triplet $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}T_1)$ in $b)$ of Fig. 4.9 (resp. $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}T_2)$ in $c)$ of Fig. 4.9), we deduce that the real scheme $1 \sqcup \langle 1 \rangle \mid 0 : 0 : 0 : 0$ (resp. $\mathcal{J} \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \mid 0 : 0 : 0 : 0$) is realizable by a real algebraic curve of class 2 (resp. 3) in a real minimal del Pezzo surface of degree 1. $\square$



a) b)

Figure 4.7:

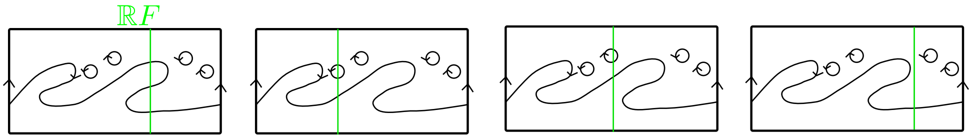
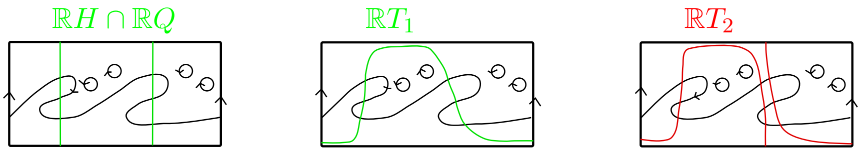


Figure 4.8:



a) b) c)

Figure 4.9: *a)* $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}H)$. *b)* After moving the hyperplane section H. *c)* Application of construction in Remark 4.3.2.

4.3.3 Harnack's construction method on Q

In Proposition 4.3.5, we realize some of the real schemes listed in Theorem 4.1.6. A variant of Harnack's construction method allows us to construct both a bidegree $(3,0)$ real algebraic maximal curve $\tilde{S}$ and a bidegree $(1,0)$ real algebraic curve T_1 on Q. Via Harnack's construction and the construction explained in Remark 4.3.2, one can construct real algebraic curves of bidegree $(3,0)$, $(1,0)$ and $(1,1)$ on Q and proves that all real schemes listed in (1) and (2) of Proposition 4.3.5 are realizable by real algebraic curves of class 2 and 3 in a real minimal del Pezzo surface of degree 1.

Proposition 4.3.5. *Let $\mathcal{S}_0 \,|\mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$ be any of the coarse real schemes listed below. Then, each real scheme $\mathcal{S}_0 \mid \mathcal{S}_{\sigma(1)} : \mathcal{S}_{\sigma(2)} : \mathcal{S}_{\sigma(3)} : \mathcal{S}_{\sigma(4)}$ is realizable, for any permutation $\sigma \in S_4$, by a real algebraic curve of class k on a real minimal del Pezzo surface of degree 1, with $k \in \{2,3\}$.*

(1) If $k=2$:

  (1) $0 \mid \alpha : \beta : \gamma : 0$, with $0 \leq \alpha+\beta+\gamma \leq 3$, and $0 \leq \alpha \leq \beta \leq \gamma$.

(2) If $k=3$:

  (1) $\mathcal{J} \mid \alpha \sqcup \langle \beta \rangle : \gamma : \delta : \varepsilon$, with $0 \leq \alpha+\beta+\gamma+\delta+\varepsilon \leq 4$, $0 \leq \gamma \leq \delta \leq \varepsilon$ and $0 \leq \alpha \leq \beta$;

  (2) $\mathcal{J} \sqcup 1 \mid \alpha \sqcup \langle \beta \rangle : 0 : 0 : 0$, with $0 \leq \alpha+\beta \leq 2$ and $0 \leq \alpha \leq \beta$;

  (3) $\mathcal{J} \mid 0 : 0 : 0 : 0$.

Proof. Fix a non-singular real algebraic curve T_1 of bidegree $(1,0)$ on Q. Pick any other real algebraic curve L_1 of bidegree $(1,0)$ on Q such that $T_1 \cap L_1$ consists of two distinct real points. Let $P_0(x,y)P_1(x,y) = 0$ be a polynomial equation defining the union of T_1 and L_1 in some local affine chart of Q. Choose 4 real lines F_i on Q, with $i = 1,2,3,4$, intersecting transversely $T_1 \cup L_1$. Replace the left side of the equation $P_0(x,y)P_1(x,y) = 0$

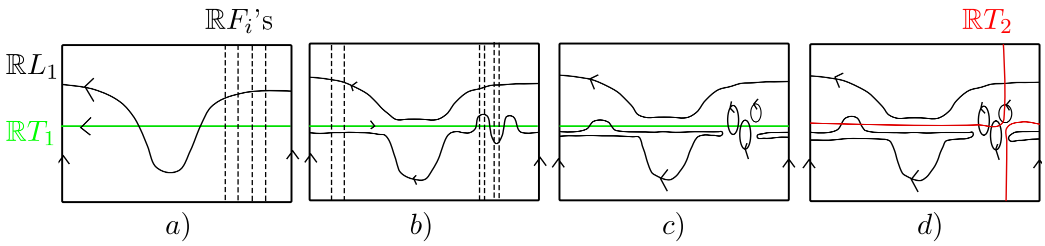


Figure 4.10: $a)-c)$ Example of Harnack's construction method. $d)$ Application of the construction in Remark 4.3.2.

with $P_0(x,y)P_1(x,y) + \varepsilon f_1(x,y)f_2(x,y)f_3(x,y)f_4(x,y)$, where $f_i(x,y) = 0$ is an affine equation of the line F_i and $\varepsilon > 0$ is a sufficient small real number. In this way one constructs a small perturbation L_2 of $T_1 \cup L_1$, where L_2 is a non-singular real maximal curve of bidegree $(2,0)$ such that $\bigsqcup_{i=1}^4 F_i \cap T_1 = L_2 \cap T_1$. In Fig. 4.10 we have depicted an example of a such construction before and after a small perturbation, respectively pictured in $a)$ and $b)$; the dashed segments are the lines F_i's.
Analogously, by a small perturbation of $T_1 \cup L_2$, one can construct a non-singular real algebraic maximal curve $\tilde{S}$ of bidigree $(3,0)$ on Q. In Fig. 4.10 we have depicted an example of a such construction before and after a small perturbation, respectively pictured in $b)$ and $c)$.
Finally, one can construct real algebraic maximal curves $\tilde{S}$ on Q and bidegree $(1,0)$, resp. $(1,1)$ thanks to Remark 4.3.2, real algebraic curves T_1, resp. T_2, on Q such that, from the arrangement of the triplets $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}T_1)$, resp. $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}T_2)$, we deduce that the real schemes in (1) above, resp. (2) above, are realizable by real algebraic curves of class 2, resp. 3, in a real minimal del Pezzo surface of degree 1. As example, from the arrangement of the triplet $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}T_1)$ in $c)$ of Fig. 4.10 (resp. $(\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}T_2)$ in $d)$ of Fig. 4.10), we deduce that the real scheme $0 \mid 1:0:1:1$ (resp. $\mathcal{J} \mid 1:1:1:1$) is realizable by a real algebraic curve of class 2 (resp. 3) in a real minimal del Pezzo surface of degree 1. □

4.3.4 Two particular constructions

With Corollary 4.3.7 we end the proof of Theorem 4.1.6. In fact, in Proposition 4.3.4, Proposition 4.3.5 and Corollary 4.3.7 we realize all the real schemes listed in Theorem 4.1.6. In this section, we give some intermediate constructions in Proposition 4.3.6 whose proofs rely on Viro's patchworking method (Section 2.2.2) and construction via dessins d'enfant (Section 2.4). In particular, we construct two real algebraic curves of bidegree $(3,0)$ on Q, whose existences directly imply Corollary 4.3.7.

Proposition 4.3.6. *There exists real algebraic curves of bidegree* $(3,0)$ *on* Q *respectively with chart as depicted in* $a)$ *and* $b)$ *of Fig. 4.11.*

Proof. We need to construct two real algebraic curves of bidegree $(2,2)$ in Σ_2 with chart as depicted in Fig. 4.11 $c)$ and $d)$. Let $\tilde{\eta}_1$, $\tilde{\eta}_2$ be trigonal $\mathcal{L}$-schemes on $\mathbb{R}\Sigma_4$ respectively as depicted in Fig. 4.12 $a)$ and $c)$. Due to Theorem 2.4.3, if the real graph associated to $\tilde{\eta}_i$ is completable in degree 4 to a real trigonal graph, then there exists a real algebraic trigonal curve $\tilde{D}_i$ realizing $\tilde{\eta}_i$. Therefore, the completion of the real graph associated to $\tilde{\eta}_i$ depicted in $b)$ of Fig. 4.12 (resp. $d)$ of Fig. 4.12) proves the existence of such $\tilde{D}_i$; see Section 2.4. Moreover, the $\tilde{D}_i$'s are reducible because they have 8 non-degenerate double points and their normalizations have 5 real connected components. In addition, the $\tilde{D}_i$'s have to be the union of a real curve of bidegree $(2,0)$ and a

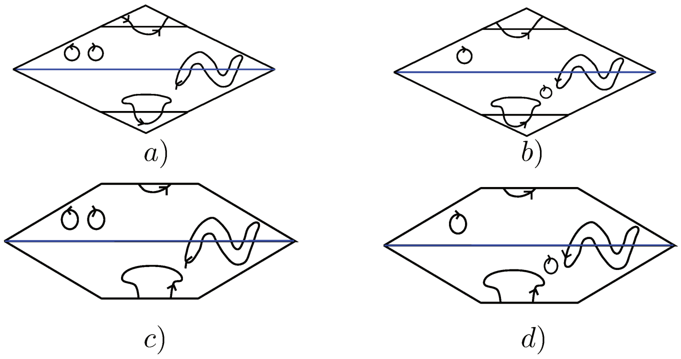


Figure 4.11: Charts of real algebraic curves.

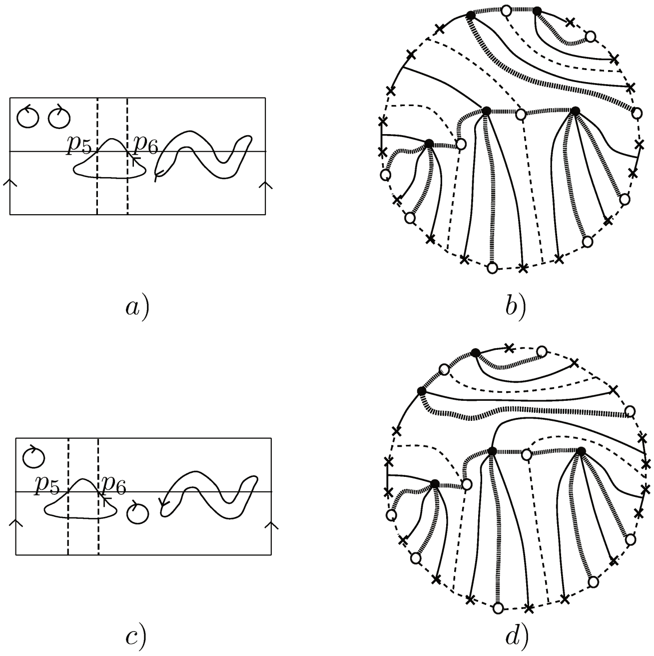


Figure 4.12: Intermediate constructions.

real curve of bidegree $(1,0)$.
Let us consider, as defined in Section 2.3, the birational transformation $\Xi := \beta_{p_5}^{-1}\beta_{p_6}^{-1} : (\Sigma_4, \tilde{D}_i) \to (\Sigma_2, D_i \cup A)$, where the points p_5, p_6, are the real double points of $\tilde{D}_i$ as depicted in $a)$ of Fig. 4.12 (resp. $c)$ of Fig. 4.12), where the dashed real fibers are those intersecting p_5 and p_6. The image via Ξ of the reducible real trigonal curve $\tilde{D}_i$ is a reducible curve, in particular the union of two non-singular real curves D_i and A, which are respectively of bidegree $(2,2)$ and $(1,0)$ in Σ_2. Moreover, the chart of D_i is as depicted in $c)$ of Fig.4.11 (resp. $d)$ of Fig.4.11).
Finally, we can apply Viro's patchworking method to the polynomials and charts of D_i and a non-singular real algebraic curve of bidegree $(1,0)$ in Σ_2, and we construct a bidegree $(3,0)$ real algebraic maximal curve on $\mathbb{C}P^3$, whose chart and arrangement with respect to the coordinates axis $\{y = 0\}$ is as depicted in $a)$ of Fig. 4.11 (resp. $b)$ of Fig. 4.11). $\square$

Propositions 4.3.6 directly implies Corollary 4.3.7 and the completion of the proof of Theorem 4.1.6.

Corollary 4.3.7. *There exist non-singular real algebraic curves of class* 2 *(resp.* 3*) in a real minimal del Pezzo surface of degree* 1*, realizing the real* *schemes* $0 \mid \langle\langle 1\rangle\rangle : 0 : 0 : 0$ *and* $0 \mid 0 : 0 : \langle\langle 1\rangle\rangle : 0$ *(resp.* $\mathcal{J} \mid \langle\langle\langle 1\rangle\rangle\rangle : 0 : 0 : 0$ *and* $\mathcal{J} \mid 0 : 0 : \langle\langle\langle 1\rangle\rangle\rangle : 0$*).*

Proof. In Proposition 4.3.6, we have constructed bidegree $(3,0)$ non-singular real algebraic maximal curves $\tilde{S}$ and also bidegree $(1,0)$ non-singular real algebraic maximal curves T_1 on Q; see the charts of $\tilde{S}$ respectively depicted in Fig. 4.11 a) and b). Therefore, there exist non-singular real algebraic curves of class 2 in a real minimal del Pezzo surface Y of degree 1 respectively realizing the real schemes $0 \mid \langle\langle 1\rangle\rangle : 0 : 0 : 0$ and $0 \mid 0 : 0 : \langle\langle 1\rangle\rangle : 0$. Applying the construction in Remark 4.3.2 to the real curves T_1, one deduces that the real schemes $\mathcal{J} \mid \langle\langle\langle 1\rangle\rangle\rangle : 0 : 0 : 0$, $\mathcal{J} \mid 0 : 0 : \langle\langle\langle 1\rangle\rangle\rangle : 0$ are also realizable by non-singular real algebraic curves of class 3 in Y. □

Chapter 5

Real algebraic curves on 4-spheres real del Pezzo surfaces of degree 2

5.1 Introduction

Let X be $\mathbb{C}P^2$ blown up at seven points in generic position; then, the surface X is a del Pezzo surface of degree 2 (see [BCC+08, pag. 289-312], [Dol12, Chapter 8]). The anti-canonical system $\phi : X \to \mathbb{C}P^2$ exhibits X as a double ramified cover of $\mathbb{C}P^2$; the branch locus of ϕ consists of an irreducible non-singular quartic Q defined by a homogeneous polynomial $f(x, y, z)$. By construction, the first Chern class $c_1(X)$ is the pull back via ϕ of the class of a line in $\mathbb{C}P^2$ ([DIK00]). Moreover, the surface X is isomorphic to the real hypersurface in $\mathbb{C}P(1, 1, 1, 2)$ defined by the weighted polynomial equation $f(x, y, z) = w^2$, with coordinates x, y, z and w respectively of weights 1 and 2. Conversely, any double cover of $\mathbb{C}P^2$ ramified along a non-singular algebraic quartic yields a del Pezzo surface of degree 2.
If one equips X with a real structure σ, the quartic Q is real and $f(x, y, z)$ can be chosen with real coefficients and so that the real surface (X, σ) is isomorphic to the real hypersurface in $\mathbb{C}P(1, 1, 1, 2)$ of equation $f(x, y, z) = w^2$. It follows that the double cover ϕ projects $\mathbb{R}X$ into the region

$$\Pi_+ := \{[X : Y : Z] \in \mathbb{R}P^2 : f(x, y, z) \geq 0\}.$$

Conversely, the double cover of $\mathbb{C}P^2$ ramified along a non-singular real quartic $Q \subset \mathbb{C}P^2$ and a choice of a real polynomial equation $f(x, y, z)$ of Q yields a real del Pezzo surface X. The surface X is $\mathbb{R}$-minimal if and only if $\mathbb{R}X$ is homeomorphic either to $\bigsqcup_{i=1}^4 S^2$ or to $\bigsqcup_{i=1}^3 S^2$. Moreover X is $\mathbb{R}$-minimal with $\mathbb{R}X$ homeomorphic to $\bigsqcup_{j=1}^4 S^2$ if and only if Q is a non-singular real maximal quartic and Π_+ is orientable; see [DK02] and, as example, Fig. 5.1.

Definition 5.1.1. *Let X be a degree 2 real minimal del Pezzo surface equipped with a real structure σ. If $\mathbb{R}X$ is homeomorphic to $\bigsqcup_{j=1}^4 S^2$, we say that X is a 4-**spheres real del Pezzo surface of degree** 2.*

Notation 5.1.2. *Let X be a 4-spheres real del Pezzo surface of degree 2. We denote the connected components of $\mathbb{R}X$ with X_1, X_2, X_3, X_4.*

The lifting of a non-singular real algebraic curve $C \subset \mathbb{C}P^2$ of degree d via ϕ is a real algebraic curve $A \subset X$ realizing $dc_1(X)$ in $H_2(X; \mathbb{Z})$. Moreover, from the topological arrangement of the triplet $(\mathbb{R}P^2, \mathbb{R}Q, \mathbb{R}C)$, one recovers the topological arrangement of the pair $(\mathbb{R}X, \mathbb{R}A)$. As example, suppose that

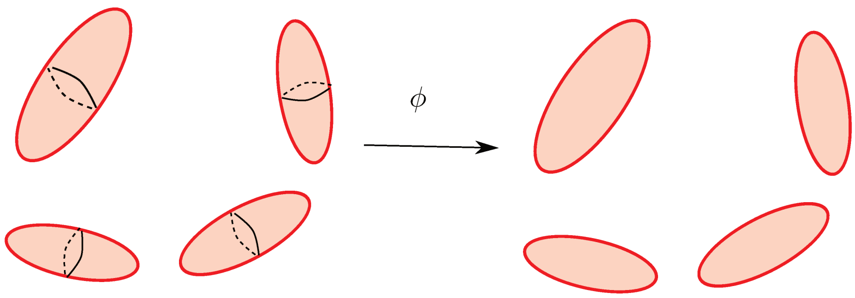


Figure 5.1: Example: $\phi : \mathbb{R}X \mapsto \Pi_+$.

Q is maximal and Π_+ is orientable (Fig. 5.1) and the triplet $(\mathbb{R}P^2, \mathbb{R}Q, \mathbb{R}C)$ is as depicted in Fig. 5.2 on the right; then the pair $(\mathbb{R}X, \mathbb{R}A)$ has topological arrangement as pictured on the left of Fig. 5.2.

Let X be a 4-spheres real del Pezzo surface of degree 2. Let $\sigma_* : H_2(X;\mathbb{Z}) \to$

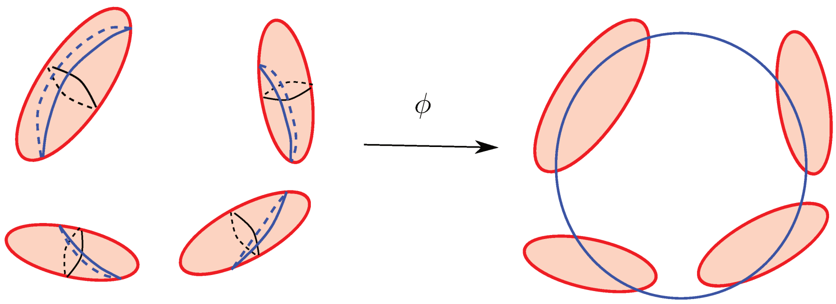


Figure 5.2: Example: Recovery of the real scheme realized by the pair $(\mathbb{R}X, \mathbb{R}A)$.

$H_2(X;\mathbb{Z})$ be the group homomorphism induced by the real structure σ on X, and let $H_2^-(X;\mathbb{Z})$ be the (-1)-eigenspace of σ_*. Since $H_2^-(X;\mathbb{Z})$ is generated by $c_1(X)$ ([BCC+08, pag. 289-312]), any real algebraic curve in X realizes $dc_1(X)$ in $H_2(X;\mathbb{Z})$, where d is some non-negative integer.

Definition 5.1.3. *Let X be a 4-spheres real del Pezzo surface of degree 2 and let $A \subset X$ be a non-singular real algebraic curve. Then, we say that A has **class** d on X if A realizes $dc_1(X)$ in $H_2(X;\mathbb{Z})$. Moreover, we call **ovals** the connected components of $\mathbb{R}A$.*

Fix a non-negative integer d. Let A be a non-singular real algebraic curve of class d in a 4-spheres real del Pezzo surface X of degree 2. In this chapter, we are interested in the classification up to homeomorphism of the pairs $(\mathbb{R}X, \mathbb{R}A)$. Due to Harnack-Klein's inequality and the adjunction formula, one obtains an immediate restriction on the number of ovals of $\mathbb{R}A$.

Proposition 5.1.4. *Let A be a real algebraic curve of class d in a 4-spheres real del Pezzo surface X of degree 2. Then, the number l of ovals of $\mathbb{R}A$ is bounded as follows:*

$$l \leq d(d-1) + 2.$$

5.1.1 Main results

Given a collection $\bigsqcup_{h=1}^{l} O_h$ of l disjoint circles embedded in S^2, the arrangement of the pair $(S^2, \bigsqcup_{h=1}^{l} O_h)$ is encoded as in Section 2.1. Let X be a 4-spheres real del Pezzo surface X of degree 2. We encode the arrangement of a given collection $\bigsqcup_{h=1}^{l} O_h$ of l disjoint circles embedded in $\mathbb{R}X$ as follows. Let

$\mathcal{S}_j$ be the codification of the arrangement of the embedded circles in X_j, with $1 \leq j \leq 4$. We say that the pair $(\mathbb{R}X, \bigsqcup_{h=1}^{l} O_h)$ realizes $\mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$.

Definition 5.1.5. *We say that $\mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$ is a **realizable real scheme in class** d, if there exist a 4-spheres real del Pezzo surface X of degree 2, and a real algebraic curve $A \subset X$ of class d, such that the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes $\mathcal{S}_1 : \mathcal{S}_2 : \mathcal{S}_3 : \mathcal{S}_4$.*

We classify in this chapter non-singular real algebraic curves of small class in a 4-spheres real del Pezzo surface of degree 2: Proposition 5.1.6 gives a complete classification for class 1 and 2; on the other hand, Theorem 5.1.7 gives a partial classification of non-singular real algebraic maximal curves of class 3. In particular, the following proposition states that Proposition 5.1.4 gives a complete system of restrictions for realizable real schemes in class 1 and 2.

Proposition 5.1.6 (Class 1 and 2)**.** *Let A be a non-singular real algebraic curve of class $d = 1, 2$ in a 4-spheres real del Pezzo surface X of degree 2. Then, the pair $(\mathbb{R}A, \mathbb{R}X)$ realizes one of the following real schemes:*

(1) if $d = 1$:

- $\alpha : \beta : 0 : 0$, *with* $0 \leq \alpha + \beta \leq 2$;

(2) if $d = 2$:

- $\alpha : \beta : \gamma : \delta$, *with* $0 \leq \alpha + \beta + \gamma + \delta \leq 4$;
- $\langle\alpha\rangle \sqcup \langle\beta\rangle : \gamma : 0 : 0$, *with* $0 \leq \alpha + \beta + \gamma \leq 2$.

Furthermore, for each real scheme $\mathcal{S}$ listed above, there exist a 4-spheres real degree 2 del Pezzo surface X and a real algebraic curve of class d in X realizing $\mathcal{S}$.

Theorem 5.1.7 (Class 3)**.** *Let A be a non-singular real algebraic maximal curve of class 3 in a 4-spheres real del Pezzo surface X of degree 2. Then, the pair $(\mathbb{R}A, \mathbb{R}X)$ realizes one of the real schemes in Table 5.1. Moreover, for each real scheme labeled with $(\circ)$ or $(\star)$, there exists a real maximal curve of class 3 in a 4-spheres real degree 2 del Pezzo surface realizing it.*

The proof of Theorem 5.1.7 relies on the proofs of Propositions 5.2.5, 5.4.1 and 5.4.14.

Remark 5.1.8. *The real scheme $2 \sqcup \langle 1\rangle \sqcup \langle 1\rangle \sqcup \langle 1\rangle$, listed in Table 5.1 of Theorem 5.1.7, is realizable by a real symplectic non-singular curve on a 4-spheres real symplectic del Pezzo surface of degree 2 (see Proposition 5.4.2).*

Remark 5.1.9. *We thank Shustin for asking the following question that highlights another direction of research on real algebraic surfaces with non-connected real parts. The classification of real schemes on real del Pezzo surfaces X_k of degree 2 (resp. of degree 1) with the real points set consisting of ($\mathbb{R}P^2$ and) $k \leq 3$ spheres, can be obtained from the statement of Theorems 5.1.7, 5.1.6 (resp. of Theorem 4.1.6) by listing the schemes that involve only k spheres. Is it possible to prove that the algebraically realization of such real schemes is an easy consequence of the proof of Theorems 5.1.7, 5.1.6 (resp. of Theorem 4.1.6)? Answering to the latter question, it may not be an easy task. But, it should be possible to give a partial answer to the symplectic version of such question exploiting real surgeries of $\mathbb{R}X_k$ along real Lagrangian spheres as explained in [Bru18].*

$5:1:1:1 \quad (\star)$	$2 \sqcup \langle 2\rangle : 3:0:0 \quad (\star)$
$4:2:1:1 \quad (\star)$	$\langle 1\rangle \sqcup \langle 2\rangle : 3:0:0 \quad (\star)$
$3:3:1:1 \quad (\star)$	$1 \sqcup \langle 1\rangle \sqcup \langle 1\rangle : 3:0:0$
$3:2:2:1 \quad (\star)$	$5 : \langle\langle 1\rangle\rangle : 0:0 \quad (\star)$
$2:2:2:2 \quad (\circ)$	$1 \sqcup \langle 3\rangle : \langle\langle 1\rangle\rangle : 0:0 \quad (\star)$
$6:1:1:0 \quad (\star)$	$2 \sqcup \langle 2\rangle : \langle\langle 1\rangle\rangle : 0:0$
$1 \sqcup \langle 4\rangle : 1:1:0 \quad (\star)$	$4:4:0:0 \quad (\star)$
$2 \sqcup \langle 3\rangle : 1:1:0$	$1 \sqcup \langle 2\rangle : 4:0:0 \quad (\star)$
$5:2:1:0 \quad (\star)$	$\langle\langle\langle 1\rangle\rangle\rangle : 4:0:0 \quad (\star)$
$1 \sqcup \langle 3\rangle : 2:1:0 \quad (\star)$	$1 \sqcup \langle 2\rangle : 1 \sqcup \langle 2\rangle : 0:0 \quad (\star)$
$2 \sqcup \langle 2\rangle : 2:1:0$	$8:0:0:0 \quad (\circ)$
$4:3:1:0 \quad (\star)$	$1 \sqcup \langle 6\rangle : 0:0:0 \quad (\star)$
$1 \sqcup \langle 2\rangle : 3:1:0 \quad (\star)$	$2 \sqcup \langle 5\rangle : 0:0:0 \quad (\circ)$
$4:2:2:0 \quad (\star)$	$3 \sqcup \langle 4\rangle : 0:0:0 \quad (\circ)$
$1 \sqcup \langle 2\rangle : 2:2:0 \quad (\star)$	$\langle 1\rangle \sqcup \langle 5\rangle : 0:0:0 \quad (\star)$
$4 : \langle\langle 1\rangle\rangle : 1:0 \quad (\star)$	$\langle 2\rangle \sqcup \langle 4\rangle : 0:0:0 \quad (\circ)$
$3:3:2:0 \quad (\star)$	$\langle 3\rangle \sqcup \langle 3\rangle : 0:0:0$
$\langle\langle 1\rangle\rangle : 3:2:0 \quad (\star)$	$1 \sqcup \langle 1\rangle \sqcup \langle 4\rangle : 0:0:0 \quad (\circ)$
$7:1:0:0 \quad (\star)$	$1 \sqcup \langle 2\rangle \sqcup \langle 3\rangle : 0:0:0$
$1 \sqcup \langle 5\rangle : 1:0:0 \quad (\star)$	$2 \sqcup \langle 1\rangle \sqcup \langle 3\rangle : 0:0:0$
$2 \sqcup \langle 4\rangle : 1:0:0 \quad (\circ)$	$2 \sqcup \langle 2\rangle \sqcup \langle 2\rangle : 0:0:0$
$3 \sqcup \langle 3\rangle : 1:0:0$	$3 \sqcup \langle 1\rangle \sqcup \langle 2\rangle : 0:0:0 \quad (\circ)$
$\langle 1\rangle \sqcup \langle 4\rangle : 1:0:0 \quad (\star)$	$4 \sqcup \langle 1\rangle \sqcup \langle 1\rangle : 0:0:0 \quad (\circ)$
$\langle 2\rangle \sqcup \langle 3\rangle : 1:0:0$	$\langle 1\rangle \sqcup \langle 1\rangle \sqcup \langle 3\rangle : 0:0:0$
$1 \sqcup \langle 1\rangle \sqcup \langle 3\rangle : 1:0:0$	$\langle 1\rangle \sqcup \langle 2\rangle \sqcup \langle 2\rangle : 0:0:0$
$1 \sqcup \langle 2\rangle \sqcup \langle 2\rangle : 1:0:0$	$1 \sqcup \langle 1\rangle \sqcup \langle 1\rangle \sqcup \langle 2\rangle : 0:0:0$
$2 \sqcup \langle 1\rangle \sqcup \langle 2\rangle : 1:0:0$	$2 \sqcup \langle 1\rangle \sqcup \langle 1\rangle \sqcup \langle 1\rangle : 0:0:0$
$3 \sqcup \langle 1\rangle \sqcup \langle 1\rangle : 1:0:0 \quad (\circ)$	$\langle 1\rangle \sqcup \langle\langle 4\rangle\rangle : 0:0:0 \quad (\circ)$
$6:2:0:0 \quad (\star)$	$\langle 2\rangle \sqcup \langle\langle 3\rangle\rangle : 0:0:0$
$1 \sqcup \langle 4\rangle : 2:0:0 \quad (\star)$	$1 \sqcup \langle 1\rangle \sqcup \langle\langle 3\rangle\rangle : 0:0:0$
$2 \sqcup \langle 3\rangle : 2:0:0 \quad (\circ)$	$1 \sqcup \langle 2\rangle \sqcup \langle\langle 2\rangle\rangle : 0:0:0$
$\langle 1\rangle \sqcup \langle 3\rangle : 2:0:0 \quad (\star)$	$1 \sqcup \langle 3\rangle \sqcup \langle\langle 1\rangle\rangle : 0:0:0$
$\langle 2\rangle \sqcup \langle 2\rangle : 2:0:0$	$2 \sqcup \langle 1\rangle \sqcup \langle\langle 2\rangle\rangle : 0:0:0$
$1 \sqcup \langle 1\rangle \sqcup \langle 2\rangle : 2:0:0$	$2 \sqcup \langle 2\rangle \sqcup \langle\langle 1\rangle\rangle : 0:0:0$
$2 \sqcup \langle 1\rangle \sqcup \langle 1\rangle : 2:0:0 \quad (\circ)$	$3 \sqcup \langle 1\rangle \sqcup \langle\langle 1\rangle\rangle : 0:0:0 \quad (\circ)$
$5:3:0:0 \quad (\star)$	$1 \sqcup \langle 1\rangle \sqcup \langle 1 \sqcup \langle 2\rangle\rangle : 0:0:0$
$1 \sqcup \langle 3\rangle : 3:0:0 \quad (\star)$	$1 \sqcup \langle 1\rangle \sqcup \langle 2 \sqcup \langle 1\rangle\rangle : 0:0:0$

Table 5.1: Real schemes in class 3

5.2 Obstructions based on Welschinger-type invariants

In Propositions 5.2.2, we prove restrictions for real algebraic curves in a 4-spheres real del Pezzo surface X of degree 2, exploiting Welschinger-type invariants and the intersection form on X. Welschinger invariants can be regarded as real analogues of genus zero Gromov-Witten invariants. They were introduced in [Wel05] and count, with appropriate signs, the real rational curves which pass through a given real collection of points in a given real rational algebraic surface. In the case of 4-spheres real del Pezzo surfaces of degree 2, the Welschinger invariants, as well as their generalizations to higher genus ([Shu14]), can be used to prove the existence of interpolating real curves of genus $0 \leq g \leq 3$; see [IKS17] and [Shu14].

Proposition 5.2.1. *[Shu14, Proposition 5] Let k be an integer greater than 1 and r_1, r_2 be two non-negative odd integers such that $r_1+r_2=2k$. Let $\mathcal{P}$ be a generic configuration of $2k+2$ real points on a 4-spheres real del Pezzo surface X of degree 2 such that X_i contains r_i points of $\mathcal{P}$, with $i=1,2$; moreover, X_3 and X_4 both contain one point of $\mathcal{P}$. Then, there exists a real algebraic curve T of class k and genus 3 in X passing through $\mathcal{P}$. Furthermore, the points of $\mathcal{P}$ belong to the one-dimensional connected components of $\mathbb{R}T$.*

Proposition 5.2.2. *Let k be an integer strictly greater than 1 and r_1, r_2 be two non-negative odd integers such that $r_1+r_2=2k$. Moreover, let A be a non-singular real algebraic curve of class d in a 4-spheres real del Pezzo surface X of degree 2. Let t denote the number of connected components of $\mathbb{R}X$ to which $\mathbb{R}A$ belongs.*

(i) Suppose that $\mathbb{R}A$ has r_i disjoint nests N_h of depth j_h on X_i, respectively with $1 \le h \le r_1$ for $i=1$, and with $r_1+1 \le h \le 2k$ for $i=2$.

(1) If $r_1, r_2 > 1$, then $\sum_{h=1}^{2k} j_h \le dk-(t-2)$;

(2) If $r_1 = 2k-1$ and $r_2 = 1$, then $\sum_{h=1}^{2k-1} j_h \le dk-(t-1)$.

(ii) Suppose that $k=2$ and $\mathbb{R}A$ has a nest N_1 of depth j_1 on X_1 and a nest N_2 of depth j_2 on X_2.

(3) Then, $j_1+j_2 \le 2d-(t-2)$.

Proof. Suppose to be under the hypothesis of (i) and suppose that r_1 and r_2 are strictly greater than 1. It follows that $\mathbb{R}A$ has at least 3 disjoint nests on X_i, with $i=1,2$. In order to prove inequality (1), let us choose a generic collection $\mathcal{P}$ of $2k+2$ real points in the following way. On each boundary of the r_1 (resp. r_2) disks in $X_1 \setminus \bigsqcup_{h=1}^{r_1} N_h$ (resp. $X_1 \setminus \bigsqcup_{h=r_1+1}^{2k} N_h$), pick a point. Moreover, pick a point on every connected component X_j, with $j=3,4$, such that the point belongs to $\mathbb{R}A$ any time the real algebraic curve has at least one oval on X_j. Then, Proposition 5.2.1 assures the existence of a real algebraic curve T of class k and genus 3 on X passing through $\mathcal{P}$. Furthermore, the points of $\mathcal{P}$ belongs to the one-dimensional connected components of $\mathbb{R}T$. Thus, the number of real intersection points of A with T is at least $2(\sum_{h=1}^{2k} j_h + (t-2))$. Inequality (1) follows directly from the fact that the intersection number $A \circ T = 2dk$ is greater or equal than the number of real intersection points of A with T.
The proof of (2) is similar to the previous one.
Suppose to be under the hypotheses of (ii). The proof of (3) does not require Proposition 5.2.1. In fact, for a given configuration $\mathcal{P}$ of 6 points in X there always exists an algebraic curve T of class 2 and genus 3 passing through $\mathcal{P}$. Moreover, if $\mathcal{P}$ consists of only real points such that at least one point of $\mathcal{P}$ lies on X_i, $\forall i \in \{1,2,3,4\}$, the curve T is real and $\mathbb{R}T$ has exactly 4 connected components. Let us choose $\mathcal{P}$ as follows. On each boundary of the 2 disks in $X_i \setminus N_i$, with $i=1,2$, pick a point. In addition, pick a point on the connected components X_j, with $j=3,4$, such that the point belongs to $\mathbb{R}A$ any time A has at least one oval on X_j. Then, there exists a real algebraic curve T of class 2 and genus 3 passing through $\mathcal{P}$ and $\mathbb{R}T$ has a connected component on each X_j, with $j=1,2,3,4$, and it has at least a connected component of dimension 1 on X_i, with $i=1,2$. Thus, the number of real intersection points of A with T is at least $2(j_1+j_2+(t-2))$. Inequality (3) follows directly from the fact

that the intersection number $A \circ T = 4d$ is greater or equal than the number of real intersection points of A with T. $\square$

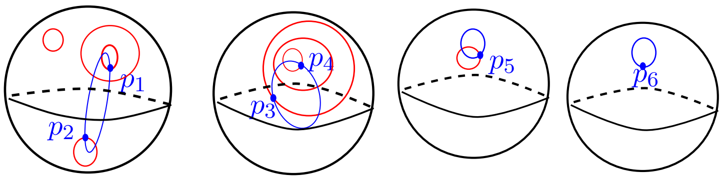


Figure 5.3: Example of unrealizable real scheme in class 3.

Example 5.2.3 (Application of Proposition 5.2.2). *Let us consider the real scheme* $\mathcal{S} := 2 \sqcup \langle 1 \rangle : 1 \sqcup \langle 1 \rangle : 1 : 0$ *in a* 4*-spheres real del Pezzo surface* X *of degree* 2*. Let us apply inequality* (3) *of Proposition 5.2.2 to prove that* $\mathcal{S}$ *is unrealizable in class* 3 *in* X*; see Fig. 5.3. Suppose that there exists a real algebraic curve* A *of class* 3 *in* X *realizing* $\mathcal{S}$*. The real scheme* $\mathcal{S}$ *has a nest of depth* 3 *on* X_1*, a nest of depth* 3 *in* X_2 *and an oval on* X_3*. Let us choose a configuration* $\mathcal{P}$ *of* 6 *real points* $p_1, .., p_6$ *as depicted in Fig. 5.3; then, there exists a real algebraic maximal curve* T *of class* 2 *in* X *passing through* $\mathcal{P}$ *and* $\mathbb{R}T$ *has a connected component on each* X_j*, with* $j = 1, 2, 3, 4$*, and it has at least a connected component of dimension* 1 *on* X_i*, with* $i = 1, 2$*; see, as example, Fig. 5.3. Inequality* (3) *implies that* 7 *has to be less or equal to* 6*; it follows that there exist no real algebraic curves of class* 3 *in* X *realizing* $\mathcal{S}$*. Finally, remark that one can not prove the unrealizability of* $\mathcal{S}$ *applying inequality* (1) *or* (2) *of Proposition 5.2.2.*

A variant of the technique used in proof of (3) of Proposition 5.2.2 leads to prohibit a particular real scheme in class 3 in a 4-spheres real del Pezzo surface of degree 2.

Lemma 5.2.4. *There are no real algebraic maximal curves of class* 3 *in a* 4*-spheres real del Pezzo surface* X *of degree* 2 *realizing the real scheme* $\mathcal{S} := \langle 1 \rangle \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle : 0 : 0 : 0$.

Proof. Suppose that there exists a non-singular real algebraic curve A of class 3 realizing $\mathcal{S}$ in X. Let us choose a configuration $\mathcal{P}$ of 6 real points as follows. On each boundary of the 4 disks in $X_1 \setminus \mathbb{R}A$, pick a point. Moreover, pick a point on the connected components X_2 and X_3. Then, there exists a non-singular real algebraic curve T of class 2 passing through $\mathcal{P}$ and T has at most two ovals on X_1 and one oval on both X_2 and X_3; see Fig. 5.4. Thus, the number of real intersection points of A with T is at least 14; see Fig. 5.4. But the intersection number $A \circ T$ is 12. $\square$

Propositions 5.1.4, 5.2.2 and Lemma 5.2.4 prove the following proposition.

Proposition 5.2.5. *Let* A *be a non-singular real algebraic maximal curve of class* 3 *in a* 4*-spheres real del Pezzo surface* X *of degree* 2*. Then, the pair* $(\mathbb{R}A, \mathbb{R}X)$ *realizes one of the real schemes in Table 5.1.*

5.3 Class 1 and 2

As explained in Section 5.1, given a non-singular real maximal quartic Q and a real curve C of degree d in $\mathbb{C}P^2$, one constructs a 4-spheres real del Pezzo

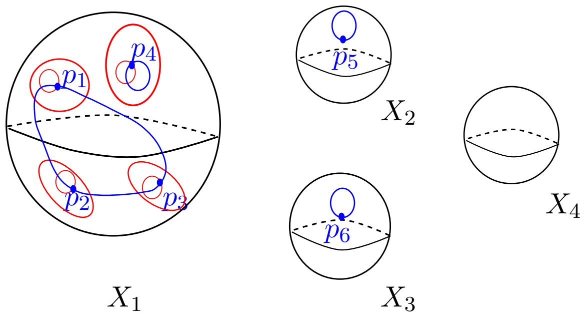
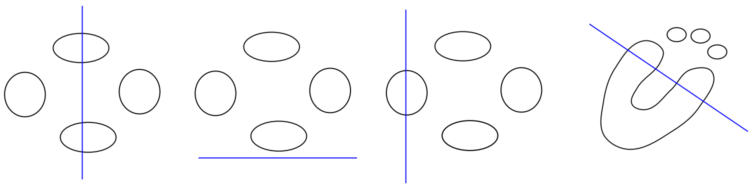


Figure 5.4: Unrealizability of $\langle 1\rangle \sqcup \langle 1\rangle \sqcup \langle 1\rangle \sqcup \langle 1\rangle : 0 : 0 : 0$ in class 3.

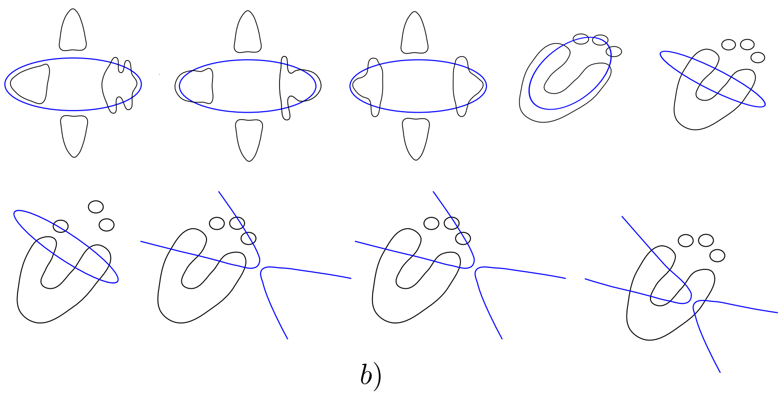
surface X of degree 2 and a real algebraic curve $A \subset X$ of class d. In addition, from the real scheme realized by the triplet $(\mathbb{R}P^2, \mathbb{R}Q, \mathbb{R}C)$, one recovers the real scheme of the pair $(\mathbb{R}X, \mathbb{R}A)$. From this fact we end the proof of Proposition 5.1.6.

Proof of Proposition 5.1.6. (Class 1 and 2). It is easy to construct real algebraic maximal quartics Q and lines (resp. conics) C in $\mathbb{C}P^2$ arranged in $\mathbb{R}P^2$ as depicted in Fig. 5.5 (resp. 5.6). From the quartics and lines (resp. conics) of Fig. 5.5 (resp. 5.6), one constructs class 1 (resp. 2) real algebraic curves realizing all real schemes listed in (1) of Proposition 5.1.6 (resp. all maximal real schemes listed in (2) of Proposition 5.1.6). The construction of real algebraic curves of class 2, realizing the other real schemes listed in (2), follows from similar constructions of conics and quartics in $\mathbb{C}P^2$. $\square$



a)

Figure 5.5: Arrangements of real lines and real maximal quartics in $\mathbb{R}P^2$.



b)

Figure 5.6: Arrangements of real maximal conics and quartics in $\mathbb{R}P^2$.

5.4 Class 3

In the proof of Proposition 5.4.1, from constructions of real maximal cubics and quartics in $\mathbb{C}P^2$ intersecting in 12 real points, we are able to realize the real schemes of Table 5.1 labeled with $(\circ)$.

Proposition 5.4.1. *For each real scheme $\mathcal{S}$ in Table 5.1 labeled with* $(\circ)$*, there exist a* 4*-spheres real degree* 2 *del Pezzo surface* X *and a real algebraic curve of class* 3 *in* X *realizing* $\mathcal{S}$*.*

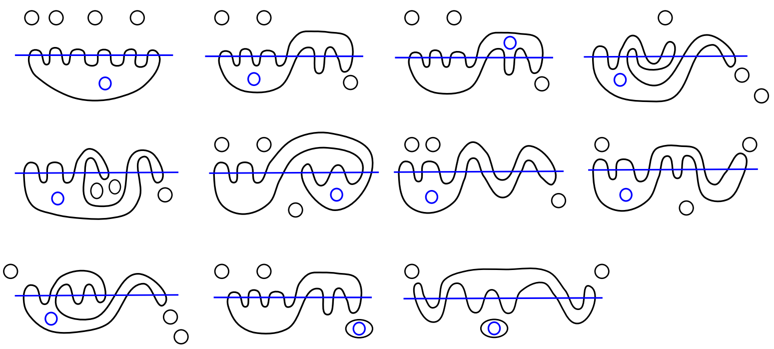


Figure 5.7: Mutual arrangements, up to isotopy, on $\mathbb{R}P^2$ of a real maximal cubic (in blue) and a real maximal quartic (in black).

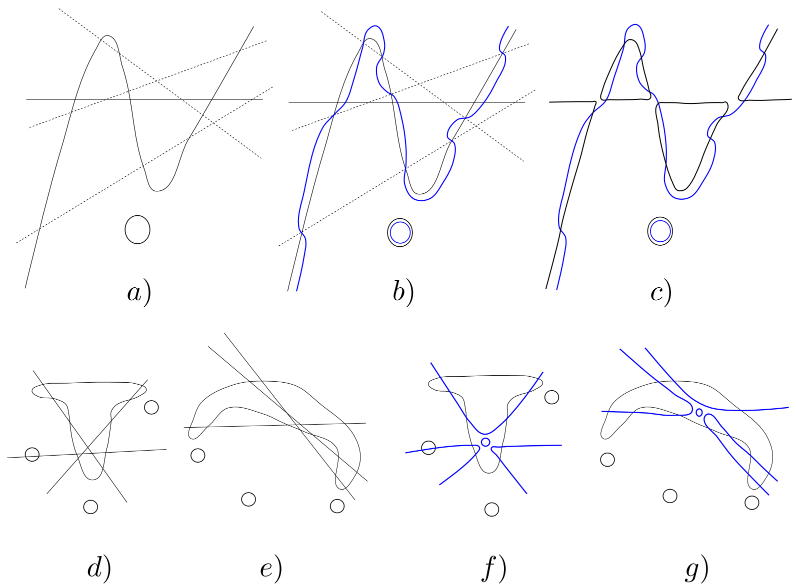


Figure 5.8: a), b), d), e) : Intermediate constructions on $\mathbb{R}P^2$. c), f), g): Mutual arrangements on $\mathbb{R}P^2$ of real maximal cubics (in blue) and a real maximal quartics (in black).

Proof. In [Ore02], Orevkov has constructed the arrangements of a real maximal quartic Q and a cubic C arranged, up to isotopy, in $\mathbb{R}P^2$ as depicted in Fig. 5.7. To each such pair corresponds a real algebraic curve of class 3 in a 4-spheres real del Pezzo surface of degree 2 realizing one of the real schemes labeled with $(\circ)$ in Table 5.1, but the real schemes $2:2:2:2$ and $3 \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle : 1:0:0$ and $2 \sqcup \langle 4 \rangle : 1:0:0$.
There exist a real cubic $\tilde{C}$ and a real line L in $\mathbb{C}P^2$ arranged in $\mathbb{R}P^2$ as represented in Fig. 5.8 a). Let $\tilde{p}(x,y,z)=0$ (resp. $l(x,y,z)=0$) be a real polynomial equation defining $\tilde{C}$ (resp. L). Pick three real lines L_1, L_2, L_3, as those depicted in dashed in Fig. 5.8 a). Replacing $\tilde{p}(x,y,z)$ with $p(x,y,z) := \tilde{p}(x,y,z)+\varepsilon l_1(x,y,z)l_2(x,y,z)l_3(x,y,z)$, where $l_i(x,y,z)$ is a real polynomial defining L_i, with $i=1,2,3$, and $\varepsilon > 0$ is a sufficient small real number, one constructs a real cubic C defined as $p(x,y,z)=0$ and arranged in $\mathbb{R}P^2$ as depicted in Fig. 5.8 b). Let $\bigcup_{i=1}^{4} L_i$ be the union of four non-real lines pairwise complex conjugated and defined by a real polynomial $u(x,y,z)$. For a sufficient

small real number $\delta > 0$, the equation $\tilde{p}(x,y,z)l(x,y,z)+\delta u(x,y,z) = 0$ defines a non-singular real plane maximal quartic Q such that $Q \cup C$ is arranged in $\mathbb{R}P^2$ as pictured in *c*) of Fig. 5.8. It follows that there exists a 4-spheres real degree 2 del Pezzo surface X and a real algebraic curve of class 3 in X realizing the real scheme $2 : 2 : 2 : 2$.
Finally, there exists a real quartic Q and three real lines arranged in $\mathbb{R}P^2$ as pictured in *d*) of Fig. 5.8 (resp. in *e*) of Fig. 5.8). Perturb the union of the three lines into a non-singular real cubic C such that $C \cup Q$ is arranged in $\mathbb{R}P^2$ as depicted in *f*) of Fig. 5.8 (resp. in *g*) of Fig. 5.8). Finally, from such pair, one constructs a real algebraic curve of class 3 in a 4-spheres real degree 2 del Pezzo surface realizing $3 \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle : 1 : 0 : 0$ (resp. $2 \sqcup \langle 4 \rangle : 1 : 0 : 0$). $\square$

5.4.0.1 Symplectic curve on a 4-spheres real symplectic degree 2 del Pezzo surface

There exists a certain mutual arrangement in $\mathbb{R}P^2$ of a real symplectic cubic and a real symplectic quartic which is unrealizable algebraically; see [Ore02]. Analogously to the algebraic case, one can construct from such arrangement in $\mathbb{R}P^2$ a real symplectic del Pezzo surface of degree 2 and a real symplectic curve of class 3 on it with topology prescribed from the arrangement on the real projective plane.

Proposition 5.4.2. *There exists a 4-spheres real symplectic degree 2 del Pezzo surface X and a non-singular real symplectic curve of class 3 in X realizing the real scheme $2 \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle : 0 : 0 : 0$.*

Proof. Let us consider $(\mathbb{C}P^2, \omega_{std}, conj)$, where ω_{std} is the symplectic Fubini-Study 2-form on $\mathbb{C}P^2$ and $conj : \mathbb{C}P^2 \to \mathbb{C}P^2$ is the standard real structure on $\mathbb{C}P^2$. Let $conj^* : H^2(\mathbb{C}P^2;\mathbb{Z}) \to H^2(\mathbb{C}P^2;\mathbb{Z})$ be the group homomorphism map induced by $conj$. It follows that $conj^*\omega_{std} = -\omega_{std}$. Due to [Ore02], there exist a non-singular real symplectic maximal quartic $\overline{Q}$ and a non-singular real symplectic maximal cubic $\overline{C}$ which are mutually arranged in $\mathbb{R}P^2$ as depicted in Fig. 5.9. The double cover $\overline{\phi} : \overline{X} \to \mathbb{C}P^2$ ramified along $\overline{Q}$ carries a natural symplectic structure ω such that $\omega = \overline{\phi}^* \omega_{std}$ ([Gro13],[Aur00]). Let c be one of the two lifts of $conj$ via the double ramified cover. Since $\overline{\phi} \circ c = conj \circ \overline{\phi}$, we have that $c^*\omega = -\omega$; namely $c : \overline{X} \to \overline{X}$ is a real structure of $\overline{X}$. Then, up to choose c, the surface $(\overline{X}, \omega, c)$ is real diffeomorphich to a 4-spheres real del Pezzo surface of degree 2 and, from $\overline{C}$, we construct a real symplectic curve of class 3 in $\overline{X}$ realizing $2 \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle : 0 : 0 : 0$. $\square$

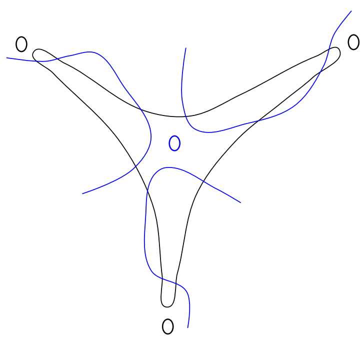


Figure 5.9:

Remark 5.4.3. *It should be possible to prove that there exists a ω-tamed almost complex structure J compatible with the action of c on X such that the class 3 real symplectic curve, constructed in Proposition 5.4.2, is J-holomorphic. The proof should follows from a variation of [Wen18, Proposition 2.2].*

5.4.1 Constructions by degeneration method and patchworking

In Sections 5.3 and 5.4, we exploit the double cover $\phi : X \to \mathbb{C}P^2$ ramified along a real non-singular quartic Q, in order to realize class $1, 2, 3$ real schemes by real algebraic curves which are *symmetric* with respect to the ramification locus of ϕ. In this section, we present a new construction method that allows to construct also *non-symmetric* real algebraic curves in X.

Remark 5.4.4. *As suggested by Brugallé, Shustin and Welschinger, it would be interesting to investigate more about real schemes which are realized by non-symmetric curves and not by symmetric ones. At the moment, via the construction method presented in this section, we can realize some of such real schemes; in fact, for each class $d \geq 5$, we can construct at least one non-symmetric real algebraic non-singular curve of class d with real scheme consisting of $2d + 1$ ovals.*

We exploit a variant of patchworking theorem (Theorem 2.2.22), proved by Shustin and Tyomkin, in order to construct non-singular real algebraic curves of class d with prescribed topology in a 4-spheres real degree 2 del Pezzo surface.
First of all, let us give some definitions. Let $Bl_{p_1,..,p_7} : S \to \mathbb{C}P^2$ be the blow-up of $\mathbb{C}P^2$ at a collection of 7 points $p_1, .., p_7$ subject to the condition that 6 of them belong to a non-singular conic in $\mathbb{C}P^2$. Then, the strict transform of the conic passing through 6 points of the collection is a smooth rational curve $E_S \subset S$ of self-intersection (-2) in S. Suppose from now on, that S contains a unique smooth rational curve of self intersection (-2). The pair (S, E_S) is called a nodal degree 2 del Pezzo pair. The anti-canonical system ϕ' of S decomposes into a regular map $S \to S'$ of degree 1 which contracts the (-2)-curve of S, and a double cover $S' \to \mathbb{C}P^2$ ramified along a quartic Q with a double point as only singularity. The surface S' is called a nodal del Pezzo surface of degree 2. Conversely, the minimal resolution of the double cover of $\mathbb{C}P^2$ ramified along a quartic with a double point as only singularity is a nodal degree 2 del Pezzo pair.
Let us equip S with a real structure σ', then E_S is real. Suppose that $\mathbb{R}S$ is homeomorphic to $\bigsqcup_{i=1}^{3} S^2$. It follows that the quartic $Q \subset \mathbb{C}P^2$ is real, it has a real non-degenerate double point as only singularity and that $\mathbb{R}Q$ consists of 3 connected components of dimension 1. Conversely, given such a quartic, one can construct a nodal degree 2 del Pezzo pair (S, E_S) where S is equipped with a real structure such that $\mathbb{R}S$ is homeomorphic to $\bigsqcup_{j=1}^{3} S^2$; see [DIK00] and see, as example, Fig. 5.10, where q (resp. p) denotes the real non-degenerate double point of the quartic Q (resp. the nodal degree 2 del Pezzo surface S').

Definition 5.4.5. *Let (S, E_S) be a nodal degree 2 del Pezzo pair. Let S be equipped with a real structure σ'. If $\mathbb{R}S$ is homeomorphic to $\bigsqcup_{j=1}^{3} S^2$, we say that (S, E_S) is a* ***3-spheres real nodal degree 2 del Pezzo pair****.*

Let X_0' be a real reducible surface given by the union of two real algebraic surfaces S and T, where

(1) T is the quadric ellipsoid;

(2) S contains a unique smooth rational (-2)-curve $E_S \subset S$ such that (S, E_S) is a 3-spheres nodal degree 2 del Pezzo pair;

(3) S and T intersect transversely along a real curve E which is a bidegree $(1, 1)$ real curve in T and E_S in S.

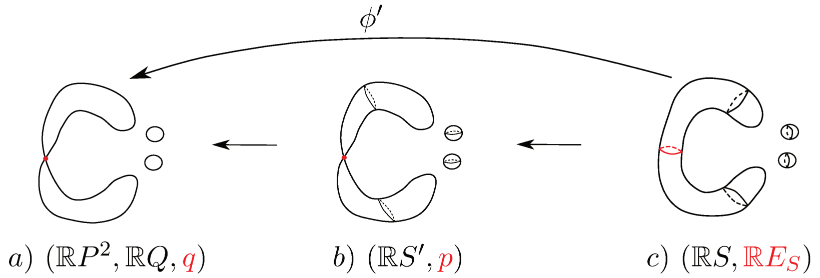


Figure 5.10: Example: Action of the anti-canonical map of a real nodal degree 2 del Pezzo pair.

Let $C_S \subset S$ and $C_T \subset T$ be non-singular real algebraic curves respectively of class $dc_1(S) - kE$ in $H_2(S;\mathbb{Z})$, and of class kE in $H_2(T;\mathbb{Z})$, such that C_T and C_S intersects along E in a real configuration $\{p_1, .., p_{2k}\}$ of $2k$ distinct points. If $\mathbb{R}E = \emptyset$, the arrangement realized by $(\mathbb{R}S \cup \mathbb{R}T, \mathbb{R}C_S \cup \mathbb{R}C_T)$ is an arrangement of ovals in $\bigsqcup_{j=1}^{4} S^2$. Otherwise, from the arrangement realized by the pair $(\mathbb{R}S \cup \mathbb{R}T, \mathbb{R}C_S \cup \mathbb{R}C_T)$, we can realize an arrangement $\mathcal{S}$ of ovals in $\bigsqcup_{j=1}^{4} S^2$ as follows. Locally $\mathbb{R}T \cap \mathbb{R}S$ is given as the intersection of two real planes as depicted in $a)$ of Fig. 5.11, and $(\mathbb{R}S \cup \mathbb{R}T) \setminus \mathbb{R}E$ has 4 connected components W_1, $W_2 \subset \mathbb{R}S$ and H_1, $H_2 \subset \mathbb{R}T$; see $b)$ of Fig. 5.11. We can glue W_1 either to H_1 or to H_2 along $\mathbb{R}E$; after making a choice, we glue W_2 to the remaining connected component along $\mathbb{R}E$; see $c)$ of Fig. 5.11. Either choices of gluing the four connected components give us the disjoint union of 4 spheres $\bigsqcup_{j=1}^{4} S^2$, and from $\mathbb{R}C_S \cup \mathbb{R}C_T$ we get an arrangement $\mathcal{S}$ of ovals in $\bigsqcup_{j=1}^{4} S^2$. See, as example, Fig. 5.12.

By Theorem 5.4.6, such topological construction is realizable algebraically.

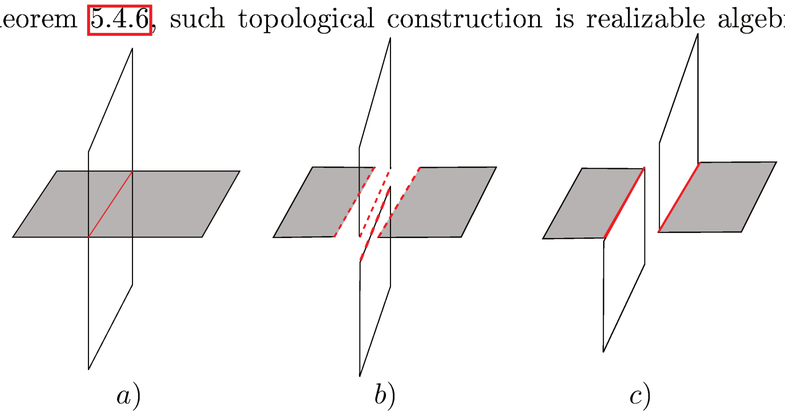


Figure 5.11:

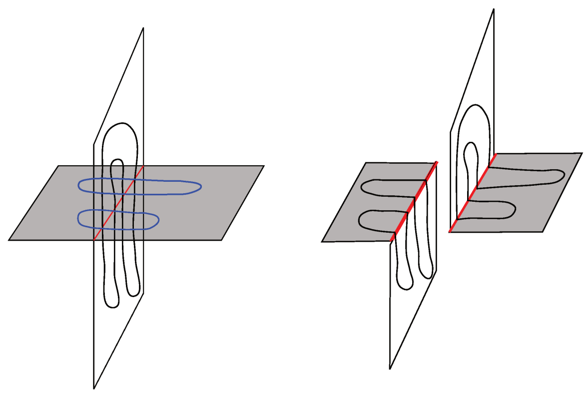


Figure 5.12:

The proof of Theorem 5.4.6 requires the existence of a real flat one-parameter family whose general fibers are (4-spheres) real degree 2 del Pezzo surfaces and whose central fiber is X_0'. We prove the existence of such family in Corollary 5.4.8.

Theorem 5.4.6. *The real scheme $\mathcal{S}$ is realizable in class d by a non-singular real algebraic curve A in a 4-spheres real degree 2 del Pezzo surface X.*

Proof. Due to Corollary 5.4.8 below, we can put X_0' in a real flat one-parameter family $\tilde{\pi}' : \mathfrak{X}' \to D(0)$, where $\mathfrak{X}'$ is a 3-dimensional real algebraic and $D(0) \subset \mathbb{C}$ is a real disk centered at 0, such that the fibers $X_t' =: \tilde{\pi}'^{-1}(t)$ are (4-spheres real) non-singular degree 2 del Pezzo surfaces, for $t \neq 0$ (and t real), and the central fiber is X_0' . By Ramanujan's Vanishing theorem ([Dol12, Section 8.3])

$$H^1(X_0'; \mathcal{O}_{X_0'}(C_0)) = 0;$$

then, Theorem 2.2.22 assures the existence of an open neighborhood $U(0) \subset D(0)$ and a deformation C_t in $\tilde{\pi}'^{-1}(t)$ such that C_t are non-singular (real) curves in X_t' for t (real) in $U(0) \setminus \{0\}$. In particular, there exists a real $\tilde{t} \in U(0) \setminus \{0\}$ such that $C_{\tilde{t}} \subset \mathbb{R}X_{\tilde{t}}'$ realizes the real scheme $\mathcal{S}$. Locally the construction of $\mathbb{R}X_{\tilde{t}}'$ from $\mathbb{R}T \cap \mathbb{R}S$ is as depicted in Fig. 5.13. As example, suppose that $\mathcal{S}$ is the real scheme $2 : 2 : 0 : 0$. Moreover, suppose that $\mathbb{R}C_S$ and $\mathbb{R}C_T$ consists of one oval respectively in $\mathbb{R}S$ and $\mathbb{R}T$ such that the arrangement of the pair $(\mathbb{R}S \cup \mathbb{R}T, \mathbb{R}C_S \cup \mathbb{R}C_T)$ is as depicted in a) of Fig. 5.14. Then, there exists a real $\tilde{t}$ such that the arrangement of the pair $(\mathbb{R}X_{\tilde{t}}', \mathbb{R}C_{\tilde{t}})$ is as depicted in b) of Fig. 5.14 and realizes $\mathcal{S}$. $\square$

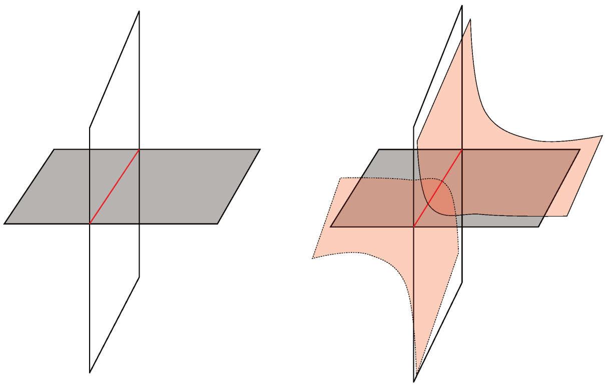


Figure 5.13:

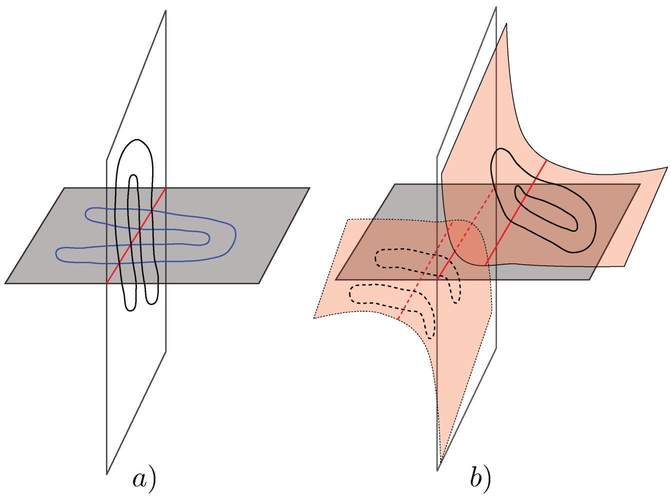


Figure 5.14:

The following proposition is used to prove Corollary 5.4.8, that we exploit in the proof of Theorem 5.4.6. In the proof we make use of a type of construction presented in [Ati58], which found recent applications in real enumerative geometry; see [BP14], [IKS17]. In particular, we prove that given a real algebraic quartic $\tilde{Q}$ in $\mathbb{C}P^2$ with a non-degenerate double point as only singularity, one can put $\tilde{Q}$ in a real flat one-parameter family in which $\tilde{Q}$ is the only singular fiber. Then, from the family of quartics one can construct a particular real flat one-parameter family of del Pezzo surfaces.

Proposition 5.4.7. *Given a real quartic $\tilde{Q}$ in $\mathbb{C}P^2$ with a real non-degenerate double point as only singularity and such that $\mathbb{R}Q$ has three connected components of dimension 1, there exists a real reducible surface X_0' equal to the union of two real algebraic surfaces $S \cup T$, where*

(1) T is the quadric ellipsoid;

(2) S is the minimal resolution of the double cover of $\mathbb{C}P^2$ ramified along $\tilde{Q}$ and it contains a unique smooth rational curve $E_S \subset S$ such that (S, E_S) is a 3-spheres nodal degree 2 del Pezzo pair;

(3) S and T intersect transversely along a curve E which is a bidegree $(1,1)$ real curve in T and the (-2)-curve E_S in S.

Proof. Let $f(x,y,z) = 0$ be a polynomial equation defining the real quartic $\tilde{Q}$ in $\mathbb{C}P^2$. Up to multiply $f(x,y,z)$ by -1, we can always put $\tilde{Q}$ in a real flat one-parameter family $\pi : \mathfrak{Q} \to D(0)$, where

- $D(0) \subset \mathbb{C}$ equipped with the standard real structure of $\mathbb{C}$, is a real disk centered at 0;
- $\mathfrak{Q} \subset \mathbb{C}P^2 \times D(0)$ is defined by $f(x,y,z) + z^4t^2 = 0$,

and such that

- the fibers $Q_t := \pi^{-1}(t)$ are non-singular (real maximal) quartic (and Π_+ is orientable) for $t \neq 0$ (and t real), and $Q_0 = \tilde{Q}$.

From the family of quartics, we can construct a real flat one-parameter family $\tilde{\pi} : \mathfrak{X} \to D(0)$ such that

- $\mathfrak{X}$ is the double cover of $\mathbb{C}P^2 \times D(0)$ ramified along $\mathfrak{Q}$ and $\mathfrak{X}$ is isomorphic to the hypersurface in $\mathbb{C}P(1,1,1,2) \times D(0)$ defined by the polynomial equation $f(x,y,z) + z^4t^2 = w^2$;
- X_0 is the double cover of $\mathbb{C}P^2$ ramified along $\tilde{Q}$. Depending on the real scheme realized by the pair $(\mathbb{R}P^2, \mathbb{R}\tilde{Q})$, the real part of X_0 is homeomorphic either to $\bigsqcup_{i=1}^2 S^2 \sqcup \{pt\}$ or to $\bigsqcup_{i=1}^2 S^2 \sqcup \vee_{j=1}^2 S^2$, where $\{pt\}$ is a point and $\vee_{j=1}^2 S^2$ is a bouquet of two 2-spheres.
- the fibers $\tilde{\pi}^{-1}(t) := X_t$ are non-singular (4-spheres real) degree 2 del Pezzo surfaces, for $t \neq 0$ (and t real).

Now, performing the blow up $Bl_p : \mathfrak{X}' \to \mathfrak{X}$ at the node p of $\mathfrak{X}$, we obtain a real flat one-parameter family $\tilde{\pi}' : \mathfrak{X}' \to D(0)$ such that $Bl_p^{-1}(p) =: T$ is the quadric ellipsoid, the fibers $\tilde{\pi}'^{-1}(t) := X_t'$ are non-singular (4-spheres real) del Pezzo surfaces of degree 2, for $t \neq 0$ (and t real), and X_0' is equal to the union of two real algebraic surfaces $S \cup T$, where S, T and E are as described in $(1) - (3)$. □

Corollary 5.4.8. *Let X_0' be a real reducible surface equal to the union of two real algebraic surfaces $S \cup T$, where*

(1) T is the quadric ellipsoid;

(2) *S contains a unique smooth rational (-2)-curve E_S such that (S, E_S) is a 3-spheres nodal degree 2 del Pezzo pair;*

(3) *S and T intersect transversely along a curve E which is a bidegree $(1,1)$ real curve in T and the (-2)-curve E_S in S.*

Then, there exists a real flat one-parameter family $\tilde{\pi}' : \mathfrak{X}' \to D(0)$, where $D(0) \subset \mathbb{C}$ is a real disk centered in 0, the fibers $\tilde{\pi}'^{-1}(t) := X'_t$ are non-singular (4-spheres real) del Pezzo surfaces of degree 2, for $t \neq 0$ (and t real), and the central fiber is X'_0.

Proof. The anti-canonical system exhibits S as the minimal resolution of the double cover of $\mathbb{C}P^2$ ramified along a real algebraic quartic $\tilde{Q}$ with a real non-degenerate double point as only singularity. Applying Proposition 5.4.7 to $\tilde{Q}$, we prove the statement. □

Remark 5.4.9. *A variation of Proposition 5.4.7 produces a reducible surface X'_0 equal to the union of two real algebraic surfaces $S \cup T$, where T is the quadric hyperboloid (resp. the real quadric with empty real part).*

5.4.1.1 Intermediate constructions: Constructions on the quadric ellipsoid and on a 3-spheres real nodal del Pezzo surface of degree 2

The aim of this section is to give some intermediate constructions that we exploit in Section 5.4.1.2 to end the proof of Theorem 5.1.7. In particular, in Section 5.4.1.2 we apply Theorem 5.4.6 and, first of all, we need to prove the existence of particular real algebraic curves in the quadric ellipsoid (Proposition 5.4.11) and on 3-spheres real nodal degree 2 del Pezzo pairs (Proposition 5.4.13).

Notation 5.4.10. *Let T be the quadric ellipsoid; see Section 3.1. In the pictures of this section, we depict $\mathbb{R}T$ (a 2-sphere) projected from some point $p \in \mathbb{R}T$ on a plane.*

In the following proposition, we exploit a variant of Harnack's construction method and some properties of three particular pencils of hyperplanes to construct real algebraic curves of bidegree $(2,2)$ and $(3,3)$ with prescribed topology in the quadric ellipsoid.

Proposition 5.4.11. *Let T be the quadric ellipsoid and let E_T be a real curve of bidegree $(1,1)$ in T. Then, for any real configuration of $2k$ distinct points in E_T fixed as follows, there exists a non-singular real algebraic maximal curve C_T of bidegree (k,k) on T, with $k = 2, 3$, intersecting transversely E_T in the $2k$ points and such that the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}C_T)$ realizes:*

- *the real scheme depicted in a) of Fig. 5.15 for $k = 2$ and 4 fixed real points;*
- *the real scheme depicted in b) of Fig. 5.15 for $k = 3$ and 6 fixed non-real points;*
- *the real scheme depicted in c) of Fig. 5.15 for $k = 3$ and 6 fixed points whose exactly 2 are real;*
- *the real scheme depicted in d) of Fig. 5.15 for $k = 3$ and 6 fixed points whose exactly 4 are real.*

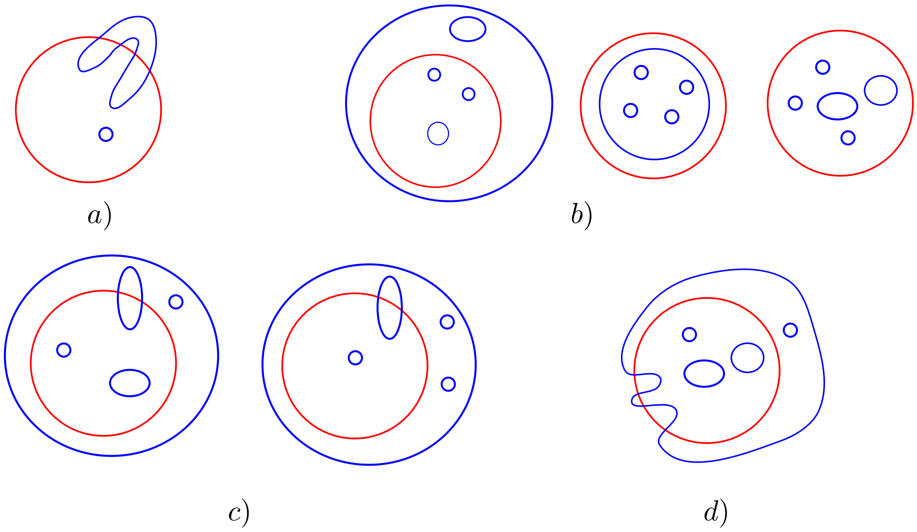


Figure 5.15:

Proof. For any two fixed distinct real points on E_T, there exists a bidegree $(1,1)$ real algebraic curve H in T passing through them. Let $P_0(x,y)P_1(x,y) = 0$ be a polynomial equation defining the union of E_T and H in T, with $x = [x_0 : x_1]$ and $y = [y_0 : y_1]$ in $\mathbb{C}P^1$. For any four fixed distinct real points on a connected component $\mathcal{E}$ of $(\mathbb{R}E_T \setminus \mathbb{R}H)$, there exist two bidegree $(1,1)$ real curves H_i in T, with $i = 1,2$, such that $H_1 \cup H_2$ passes through the fixed four points. Replace the left side of the equation $P_0(x,y)P_1(x,y) = 0$ with $P_0(x,y)P_1(x,y) + \varepsilon f_1(x,y)f_2(x,y)$, where $f_i(x,y) = 0$ is an equation for H_i and $\varepsilon > 0$ is a sufficient small real number. In this way one constructs a small perturbation $\tilde{H}$ of $E_T \cup H$, where $\tilde{H}$ is a bidegree $(2,2)$ non-singular real curve such that $\bigcup_{i=1}^{2} H_i \cap E_T = \tilde{H} \cap E_T$. The explained construction method allows to construct a bidegree $(2,2)$ real algebraic curve B in T realizing the real scheme depicted in $a)$ of Fig. 5.15. Analogously, for a real configuration of six fixed distinct points on $E_T \setminus \tilde{H}$ such that exactly 2 points (resp. 4 points) are real and belong to the same connected component $\mathcal{F}$ of $(\mathbb{R}E_T \setminus \mathbb{R}\tilde{H})$, one can construct bidegree $(3,3)$ non-singular real algebraic curves C_T in T passing through the six fixed points and realizing the real scheme on the left depicted in $b)$ of Fig. 5.15 (resp. all real schemes in $c)$ and $d)$ of Fig. 5.15).
Given a real configuration of 6 non-real points $p_1, \overline{p}_1, p_2, \overline{p}_2, p_3, \overline{p}_3$ on E_T, with $p_i = (x_i, y_i)$ and $\overline{p}_i = (\overline{y}_i, \overline{x}_i)$, where x_i, y_i are in $\mathbb{C}P^1$ and $\overline{x}_i, \overline{y}_i$ are the image of x_i, y_i via the standard complex conjugation on $\mathbb{C}P^1$. Let Π_i be a pencil of hyperplanes in T with base points p_i and $\overline{p}_i$, with $i = 1,2,3$. We want to show that one can always construct a non-singular real algebraic curve C_T of bidegree $(3,3)$ as perturbation of the union of three hyperplanes respectively of Π_1, Π_2 and Π_3 such that the arrangement of the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}C_T)$ is as depicted on the left (resp. on the right) of $b)$ in Fig. 5.16. Namely, we prove in the following that one can always find three hyperplanes respectively of Π_1, Π_2 and Π_3 whose union and real arrangement with respect to $\mathbb{R}E_T$ is as depicted on the left (resp. on the right) of $a)$ in Fig. 5.16.
First of all, remark that on each of the two connected components of $\mathbb{R}T \setminus \mathbb{R}E_T$, the real part of the real hyperplanes of the pencil Π_i vary from a real point q_i to $\mathbb{R}E_T$, with $i = 1,2,3$. Moreover, the real points q_1, q_2 and q_3 are distinct points. There exist two real hyperplanes $H_j \subset \Pi_j$ and $H_k \subset \Pi_k$ such that they are tangent in a real point s_{jk}, they do not contain q_i and their real part is as depicted in $a)$ of Fig. 5.17 (resp. in $e)$ of Fig. 5.17). Pick the real hyperplane $H_i \subset \Pi_i$ passing through s_{jk}. Then, the real part of $H_j \cup H_k \cup H_i$ is as depicted in $b)$ of Fig. 5.17 (resp. in $f)$ of Fig. 5.17). It follows that there

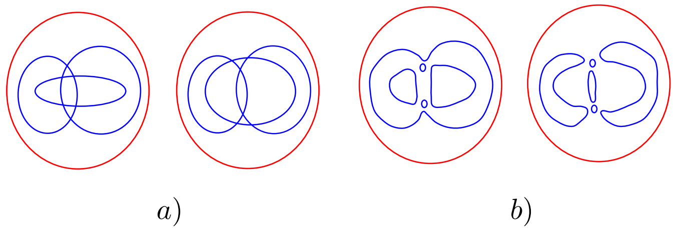


Figure 5.16: $\mathbb{R}E_T$ in red.

exists a real hyperplane of the pencil Π_i whose real arrangement with respect to $\mathbb{R}H_j \cup \mathbb{R}H_k$ is as depicted in c) of Fig. 5.17 (resp. in g) of Fig. 5.17). In conclusion, a small perturbation of the union of such three hyperplanes has real part as depicted in d) of Fig. 5.17 (resp. in h) of Fig. 5.17). □

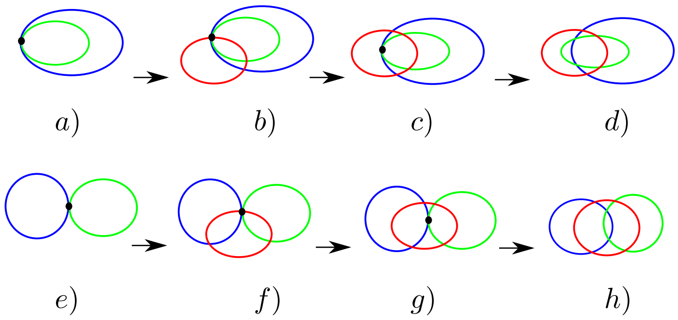


Figure 5.17:

In Proposition 5.4.13 we construct particular real algebraic curves in 3-spheres real nodal del Pezzo pairs of degree 2 via the anti-canonical map. In order to accomplish a particular construction in the proof of Proposition 5.4.13, we need the following lemma.

Lemma 5.4.12. *There exist real algebraic curves Q and C respectively of degree 4 and 3 in $\mathbb{C}P^2$ with a unique real non-degenerate double singularity at a point q, such that the triplet $(\mathbb{R}\Sigma_1, \mathbb{R}Q, \mathbb{R}C)$ realizes the real scheme depicted in a) of Fig. 5.18 (resp. in b) of Fig. 5.18, resp. in c) of Fig. 5.18).*

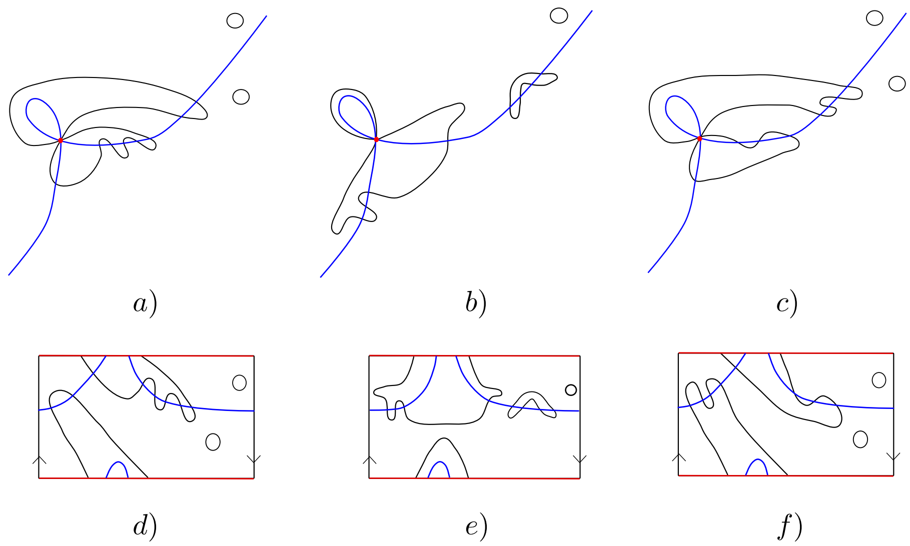


Figure 5.18:

Proof. The blow-up of $\mathbb{C}P^2$ at the point q is the first Hirzebruch surface Σ_1 (Section 2.3). Then, in order to prove the statement, it is sufficient to construct a reducible real algebraic curve K_1 (resp. K_2, resp. K_3) of bidegree $(3,4)$ in Σ_1 as union of two non-singular real algebraic curves $\tilde{Q}$ and A respectively of bidegree $(2,2)$ and $(1,2)$ in Σ_1 such that the triplet $(\mathbb{R}\Sigma_1, \mathbb{R}\tilde{Q}, \mathbb{R}A)$ realizes the $\mathcal{L}$-scheme depicted in $d)$ of Fig. 5.18 (resp. in $e)$ of Fig. 5.18, resp. in $f)$ of Fig. 5.18).

Let $\tilde{\eta}_1$, $\tilde{\eta}_2$ and $\tilde{\eta}_3$ be trigonal $\mathcal{L}$-schemes in $\mathbb{R}\Sigma_5$ respectively as depicted in

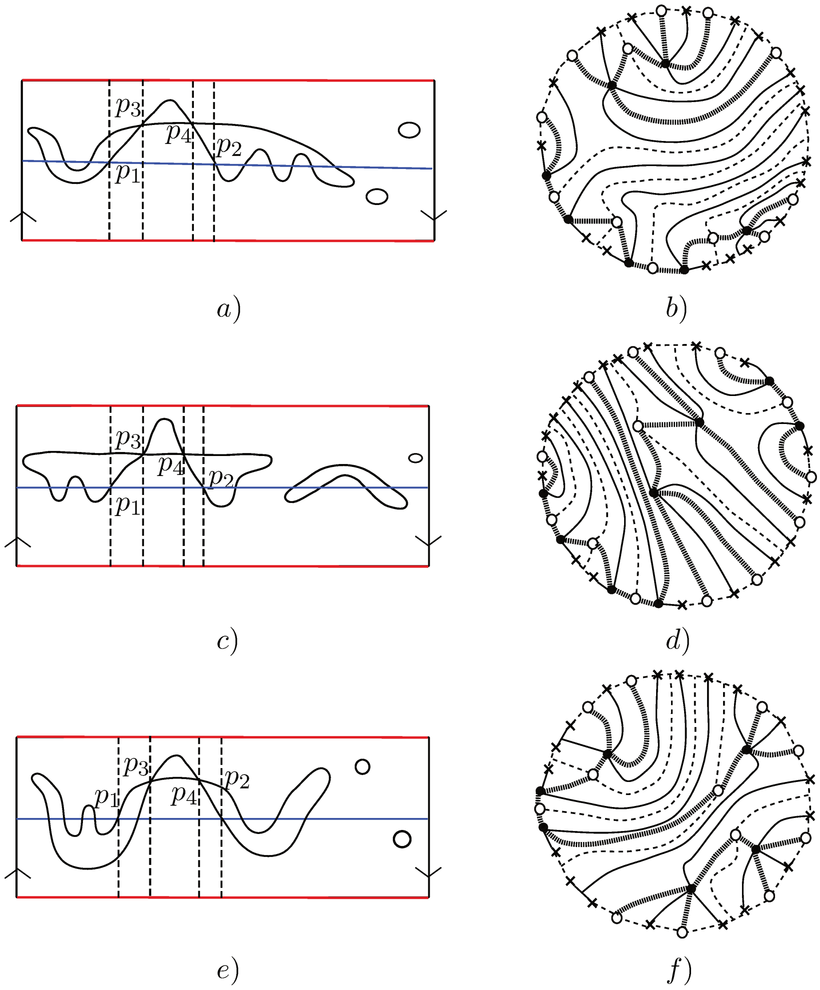


Figure 5.19: Intermediate constructions.

$a)$, $c)$ and $e)$ of Fig. 5.19. Due to Theorem 2.4.3, if the real graph associated to $\tilde{\eta}_i$ is completable in degree 5 to a real trigonal graph, then there exists a real algebraic trigonal curve $\tilde{K}_i$ realizing $\tilde{\eta}_i$, for $i = 1, 2, 3$. Therefore, the completion of the real graph associated to $\tilde{\eta}_i$ depicted respectively in $b)$, for $i = 1$, in $d)$, for $i = 2$, and in $f)$, for $i = 3$, of Fig. 5.19, proves the existence of such $\tilde{K}_i$; see Section 2.4. The trigonal curve $\tilde{K}_i$ is reducible because it has 12 non-degenerate double points and its normalization has 4 real connected components. In particular $\tilde{K}_i$ has to be the union of a real curve of bidegree $(2,0)$ and a real curve of bidegree $(1,0)$.

Let us consider the birational transformation $\Xi := \beta_{p_1}^{-1}\beta_{p_2}^{-1}\beta_{p_3}^{-1}\beta_{p_4}^{-1} : (\Sigma_5, \tilde{K}_i) \to (\Sigma_1, K_i)$, defined as in Section 2.3, where the points p_j's, with $j = 1, 2, 3, 4$, are the real double points of $\tilde{K}_i$ as depicted in $a)$ of Fig. 5.19 (resp. $c)$ of Fig. 5.19, resp. $e)$ of Fig. 5.19), and the dashed real fibers are those intersecting the p_j's. The image via Ξ of the reducible real trigonal curve $\tilde{K}_i$ is a reducible curve K_i of bidegree $(3,4)$ which is the union of two non-singular real curves $\tilde{Q}$ and A, respectively of bidegree $(2,2)$ and $(1,2)$ in Σ_1. Moreover, the $\mathcal{L}$-scheme of K_1 (resp. K_2, resp. K_3) is as depicted in a) of Fig. 5.18 (resp. b) of Fig. 5.18, resp. c) of Fig. 5.18). □

Proposition 5.4.13. *There exist 3-spheres real nodal del Pezzo pairs (S, E_S) and non-singular real algebraic curves $C_S \subset S$ realizing the class $3c_1(S) - kE_S$ in $H_2(S, \mathbb{Z})$ and such that the triplet $(\mathbb{R}S, \mathbb{R}E_S, \mathbb{R}C_S)$ is arranged:*

1. *as depicted in Fig. 5.20, for $k = 2$;*
2. *as depicted in Fig. 5.21 and in Fig. 5.22, for $k = 3$.*

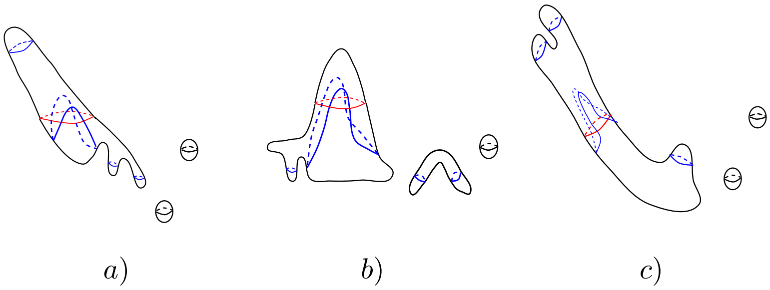


Figure 5.20: $\mathbb{R}E_S$ in red.

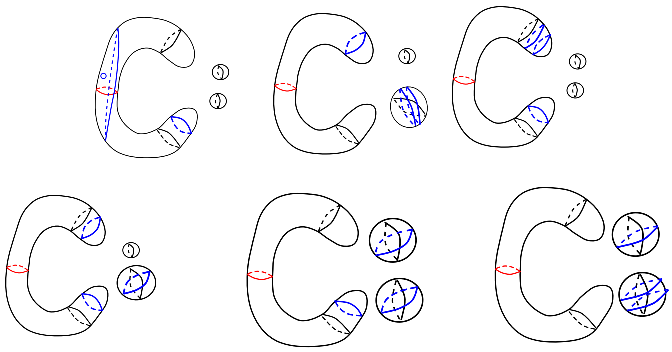


Figure 5.21: $\mathbb{R}E_S$ in red.

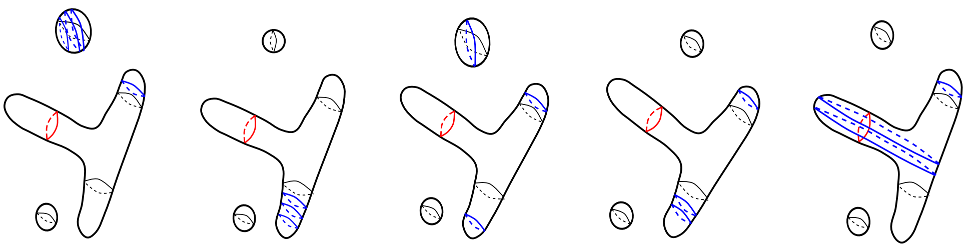
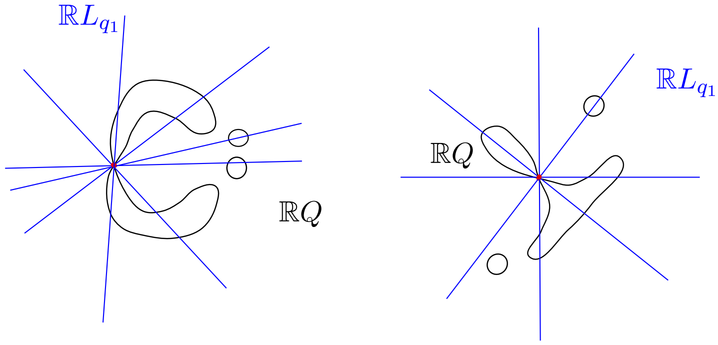


Figure 5.22: $\mathbb{R}E_S$ in red.

Proof. Let $\tilde{Q}$ be a real quartic with a real non-degenerate and non-isolated double point at q as only singularity and such that $\mathbb{R}\tilde{Q}$ has three connected components; let C be a real curve of degree d with one k-fold singularity at q. To a pair $(\tilde{Q}, C)$ correspond a 3-spheres real nodal del Pezzo pair (S, E_S) and a real algebraic curve $C_S \subset S$ of class $dc_1(S) - kE_S$ in $H_2(S, \mathbb{Z})$ with topology described by the topology of $(\mathbb{R}P^2, \mathbb{R}\tilde{Q}, \mathbb{R}C)$.
Firs of all, Lemma 5.4.12 immediately implies the existence of a 3-spheres real



a) *b*)

Figure 5.23:

nodal degree 2 del Pezzo pair (S, E_S) and a curve $C_S \subset S$ of class $3c_1(S) - 2E_S$ such that the triplet $(\mathbb{R}S, \mathbb{R}E_S, \mathbb{R}C_S)$ is arranged as depicted in Fig. 5.20. Finally, there exists a real plane quartic $\tilde{Q}_1$ with a real non-degenerate double point q_1 as only singularity and a pencil of lines $L_{q_1} \subset \mathbb{C}P^2$, centered at q_1, such that $\mathbb{R}\tilde{Q}_1 \cup \mathbb{R}L_{q_1}$ are arranged as depicted in a) of Fig. 5.23 (resp. in b) of Fig. 5.23). For every fixed point, distinct from q_1, there exists a unique line of the pencil L_{q_1} passing through it. Therefore, given three distinct real points, different from q_1, we fix three lines of L_{q_1} and their union is a cubic C_1 with a triple point at q_1. From $\tilde{Q}_1$ and C_1 one can construct 3-spheres real nodal degree 2 del Pezzo pairs (S, E_S) and curves $C_S \subset S$ of class $3c_1(S) - 3E_S$ such that the triplet $(\mathbb{R}S, \mathbb{R}E_S, \mathbb{R}C_S)$ is arranged as depicted in Fig. 5.21 (resp. Fig. 5.22). □

5.4.1.2 Final constructions

We construct non-singular real algebraic curves of class 3 in 4-spheres real del Pezzo surfaces of degree 2 realizing the real schemes labeled with $(\star)$ in Table 5.1. The proof is going to combine the results and constructions of Theorem 5.4.6 and Propositions 5.4.11, 5.4.13.

Proposition 5.4.14. *For each real scheme $\mathcal{S}$ labeled with $(\star)$ in Table 5.1, there exist a 4-spheres real del Pezzo surface X of degree 2 and a real algebraic curve of class 3 in X realizing $\mathcal{S}$.*

Proof. Pick the 3-spheres real nodal degree 2 del Pezzo pair (S, E_S) and a real algebraic curve $C_S \subset S$ of class $3c_1(X) - 2E_S$ constructed in Proposition 5.4.13. Due to Corollary 5.4.8, there exist a real algebraic surface X_0' as union of S and T, intersecting along E, and a real algebraic curve C_0 as union of C_S and C_T, intersecting along $2k$ points of E; where T is the quadric ellipsoid and $C_T \subset T$ the real algebraic curve of bidegree $(2,2)$ constructed in Proposition 5.4.11. Then, thanks to Theorem 5.4.6, there exist a 4-spheres real degree 2 del Pezzo surface X and a non-singular real algebraic curve $A \subset X$ of class 3 such that the arrangement of the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes the real scheme $2 \sqcup \langle 2 \rangle : 3 : 0 : 0$ (resp. $1 \sqcup \langle 2 \rangle : 2 : 2 : 0$, resp. $4 : 4 : 0 : 0$). See Example 5.4.15 (1).
In order to realize the remaining real schemes labeled with $(\star)$ in Table 5.1, we apply the previous construction as follows. Firs of all, pick a 3-spheres real nodal degree 2 del Pezzo pair (S, E_S) and a real algebraic curve $C_S \subset S$ of class $3c_1(X) - 3E_S$ constructed in Proposition 5.4.13. Due to Corollary 5.4.8, there exist a real algebraic surface X_0' as union of S and T, intersecting along E, and a real algebraic curve C_0 as union of C_S and C_T, intersecting along $2k$ points of E; where T is the quadric ellipsoid and $C_T \subset T$ a real

algebraic curve of bidegree $(3,3)$ constructed in Proposition 5.4.11. Then, thanks to Theorem 5.4.6, for each real scheme $\mathcal{S}$ labeled with $(\star)$ in Table 5.1, there exist a 4-spheres real degree 2 del Pezzo surface X and a non-singular real algebraic curve $A \subset X$ of class 3 such that the arrangement of the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes $\mathcal{S}$. See Example 5.4.15 (2). $\square$

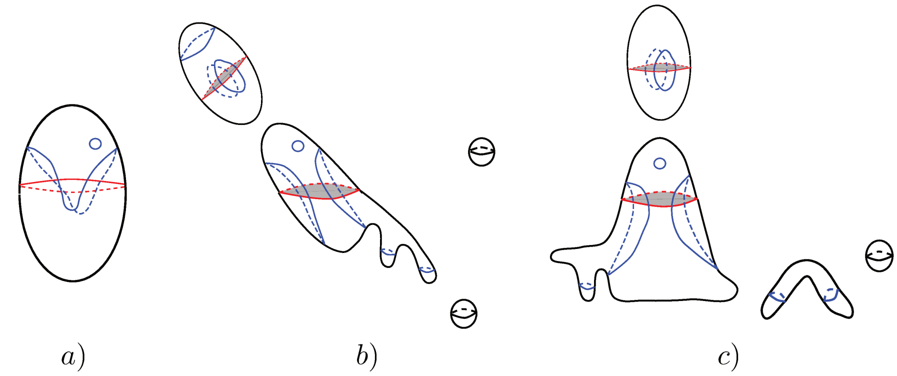


Figure 5.24: $\mathbb{R}E$ in red.

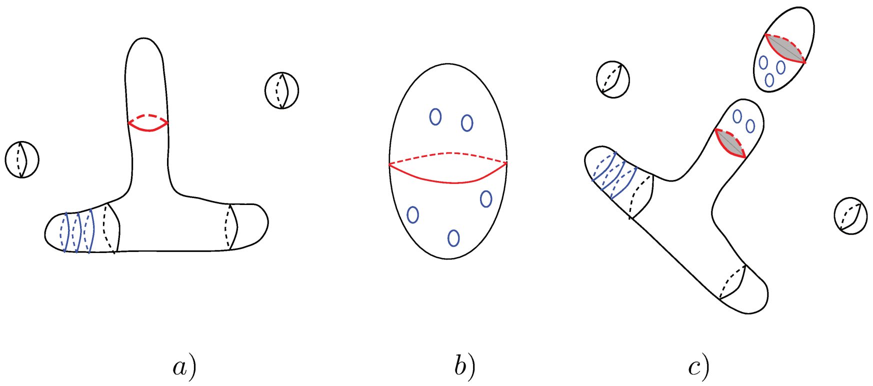


Figure 5.25: $\mathbb{R}E$ in red.

Example 5.4.15. *(1) We construct a real algebraic curve of class* 3 *in a* 4*-spheres real del Pezzo surface of degree* 2 *as in proof of Proposition 5.4.14, with real scheme* $2 \sqcup \langle 2 \rangle : 3 : 0 : 0$ *(resp.* $1 \sqcup \langle 2 \rangle : 2 : 2 : 0$*).*

- *Let* $C_S \subset S$ *be the real algebraic curve such that* $\mathbb{R}E_S \cup \mathbb{R}C_S$ *is arranged in* $\mathbb{R}S$ *as pictured in a) of Fig. 5.20 (resp. b) of Fig. 5.20).*
- *Let* $C_T \subset T$ *be the real algebraic curve of bidegree* $(2,2)$ *such that* $\mathbb{R}E_T \cup \mathbb{R}C_T$ *is arranged in* $\mathbb{R}T$ *as depicted in a) of Fig. 5.24.*
- *Thanks to Theorem 5.4.6, there exists a real algebraic curve* A *of class* 3 *in a real del Pezzo surface* X *of degree* 2 *realizing the real scheme* $2 \sqcup \langle 2 \rangle : 3 : 0 : 0$ *(resp.* $1 \sqcup \langle 2 \rangle : 2 : 2 : 0$*). See b) of Fig. 5.24 (resp. c) of Fig. 5.24).*

(2) We construct a real algebraic curve of class 3 *in a* 4*-spheres real del Pezzo surface of degree* 2 *as in proof of Proposition 5.4.14, with real scheme* $\langle 1 \rangle \sqcup \langle 2 \rangle : 3 : 0 : 0$.

- *Let* $C_S \subset S$ *be the real algebraic curve such that* $\mathbb{R}E_S \cup \mathbb{R}C_S$ *is arranged in* $\mathbb{R}S$ *as pictured in a) of Fig. 5.25.*
- *Let* $C_T \subset T$ *be the real algebraic curve of bidegree* $(3,3)$ *such that* $\mathbb{R}E_T \cup \mathbb{R}C_T$ *is arranged in* $\mathbb{R}T$ *as depicted in b) of Fig. 5.25.*

- *Thanks to Theorem 5.4.6, there exists a real algebraic curve A of class 3 in a real del Pezzo surface X of degree 2 realizing the real scheme $\langle 1 \rangle \sqcup \langle 2 \rangle : 3 : 0 : 0$. See c) Fig. 5.25.*

Remark 5.4.16. *A real degree 2 del Pezzo surface X is $\mathbb{R}$-minimal if and only if its real part is homeomorphic either to $\bigsqcup_{i=1}^{4} S^2$ or to $\bigsqcup_{i=1}^{3} S^2$. In the latter case, we say that X is a a 3-spheres real del Pezzo surface of degree 2. Using the variation proposed in Remark 5.4.9, it is possible to exploit a construction similar to the one presented in Proposition 5.4.14 in order to realize real algebraic curves of class d with prescribed topology in a 3-spheres real del Pezzo surface of degree 2.*

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Index of definitions and notations

université PARIS-SACLAY
École doctorale de mathématiques Hadamard (EDMH)

Titre : Courbes algébriques réelles dans les surfaces de del Pezzo réelles minimales.

Mots clés : classification des types topologiques, courbes algébriques réelles, surfaces rationnelles réelles minimales, 16ème problème de Hilbert, surfaces de del Pezzo, quadric ellipsoïde.

Résumé : L'étude topologique des variétés algébriques réelles remonte au moins aux travaux de Harnack, Klein, et Hilbert au 19éme siecle ; en particulier, la classification des types d'isotopie réalisés par les courbes algébriques réelles d'un degré fixé dans $\mathbb{R}P^2$ est un sujet qui a connu un essor considérable jusqu'à aujourd'hui. En revanche, en dehors des études concernants les surfaces de Hirzebruch et les surfaces de degré au plus 3 dans $\mathbb{R}P^3$, à peu près rien n'est connu dans le cas de surfaces ambiantes plus générales. Cela est dû en particulier au fait que les variétés construites en utilisant le "patchwork" sont des hypersurfaces de variétés toriques. Or, il existe de nombreuses autre surfaces algébriques réelles. Parmi celles-ci se trouvent les surfaces rationnelles réelles, et plus particulièrement les surfaces $\mathbb{R}$-minimales. Dans cette thèse, on élargit l'étude des types d'isotopie réalisés par les courbes algébriques réelles aux surfaces réelles minimales de del Pezzo de degré 1 et 2. En outre, on termine la classification des types topologiques réalisés par les courbes algébriques réelles séparantes et non-séparantes de bidegré $(5,5)$ sur la quadrique ellipsoïde.

Title : Real minimal curves on real minimal del Pezzo surfaces.

Keywords : classification of topological types, real algebraic curves, real minimal rational surfaces, Hilbert's 16th problem, del Pezzo surfaces, quadric ellipsoid.

Abstract : The study of the topology of real algebraic varieties dates back to the work of Harnack, Klein and Hilbert in the 19th century; in particular, the isotopy type classification of real algebraic curves with a fixed degree in $\mathbb{R}P^2$ is a classical subject that has undergone considerable evolution. On the other hand, apart from studies concerning Hirzebruch surfaces and at most degree 3 surfaces in $\mathbb{R}P^3$, not much is known for more general ambient surfaces. In particular, this is because varieties constructed using the patchworking method are hypersurfaces of toric varieties. However, there are many other real algebraic surfaces. Among these are the real rational surfaces, and more particularly the $\mathbb{R}$-minimal surfaces. In this thesis, we extend the study of the topological types realized by real algebraic curves to the real minimal del Pezzo surfaces of degree 1 and 2. Furthermore, we end the classification of separating and non-separating real algebraic curves of bidegree $(5,5)$ in the quadric ellipsoid.

Université Paris-Saclay
Espace Technologique / Immeuble Discovery
Route de l'Orme aux Merisiers RD 128 / 91190 Saint-Aubin, France